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Classification of coupled systems with two-component nonlinear diffusion equations by the invariant subspace method

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Abstract
The invariant subspace method is developed to perform classification of systems with two-component nonlinear diffusion equations, which was carried out with respect to the invariant subspaces $W^1_{n_1} \times W^2_{n_2}$ defined by linear ordinary differential equations. As a result, the corresponding exact solutions generated by invariant subspaces to the resulting systems are obtained. In most cases, two components of these exact solutions belong to different 'scalar' subspaces. Behaviour to several exact solutions of the systems is described.

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1. Introduction

In this paper, we provide a classification to systems with two-component nonlinear diffusion equations

$$\begin{aligned} u_t &= [f(u,v)u_x + p(u,v)v_x]_x + r(u,v) \equiv F_1[u,v], \\ v_t &= [g(u,v)v_x + q(u,v)u_x]_x + s(u,v) \equiv F_2[u,v], \end{aligned} \tag{1}$$

based on the invariant subspaces defined by linear ordinary differential equations (ODEs). System (1) can be regarded as a generalization of the celebrated nonlinear diffusion equation with source term

$$u_t = (f(u)u_x)_x + g(u).$$

Such systems with diffusion terms have many physical and biological applications. For instance, it describes the diffusion of impurities in semiconductors [1], models contact inhibition between cell population [2] and two spatially distributed populations in a predator–prey relationship with each other [3–5], etc.

Various invariant subspaces to nonlinear evolution equations of the form

$$u_t = f(u, u_x)u_{xx} + g(u, u_x)$$

for certain f and g have been obtained (see [6, 7] and references therein), which yield a number of exact solutions of the equations. In [6], King constructed a class of non-group-invariant solutions to a number of nonlinear diffusion equations. Those solutions are associated with the invariant subspaces admitted by the equations or their variant forms, which play an important role in the study of the behaviour of general solutions to the equations. In [8] and in the previous preprints, Galaktionov has suggested the invariant subspace method and then utilized it to obtain exact solutions of nonlinear evolution equations with quadratic nonlinearities. For instance, it was shown that exact solutions of the quasilinear heat equations

$$u_t = \left(u^{-\frac{4}{3}}u_x\right)_x - au^{-\frac{1}{3}} + bu^{\frac{7}{3}} + cu \equiv \bar{F}_1[u] \quad (u > 0, a, b, c \in \mathbb{R})$$

can be constructed on the linear subspaces of polynomial or trigonometric form, which are admitted by the spatial operator $\bar{F}_1[u]$. The corresponding nonlinear evolution equations on invariant subspaces are shown to be equivalent to finite-dimensional dynamic systems. A systematic approach was developed long before book [7]. Book [7] contains a systematic account of the approach, including some new results and many applications. Indeed, by the logarithmic change of variables the fundamental solution of the standard heat equation

$$u_t = \triangle u \qquad \text{in} \quad \mathbb{R}^N \times \mathbb{R}_+ \left(\triangle = \sum_{i=1}^{N} \frac{\partial^2}{\partial x^2}\right)$$

leads to the exact solution of the semi-linear parabolic equation

$$v_t = \triangle v + |\bigtriangledown v|^2 \equiv \bar{F}_2[v]$$

on the 2D subspace $W_2 = \mathcal{L}\{1, |x|^2\}$ preserved by the operator $\bar{F}_2$. It was also shown that the N-solitons of integrable equations derived by the Baker–Hirota's bilinear method belong to a linear subspace of exponential functions in the sense of change of variables. A number of nonlinear differential operators possessing invariant subspace of different dimension were described in [7–14]. So many different types of solutions to nonlinear evolution equations can be obtained through the invariant subspace method.

Let us give a brief account of the invariant subspace method as is presented in [7]. Consider the general evolution equation

$$u_t = F[u], \tag{2}$$

where F is a kth-order ordinary differential operator and $F[u] \equiv F(x, u, u_x, u_{xx}, \ldots)$ is a given sufficiently smooth function of the indicated variables. Let $\{f_i(x),\ i = 1, \ldots, n\}$ be a finite set of $n \geqslant 1$ linearly independent functions, and W_n denote their linear span $W_n = \mathcal{L}\{f_1(x), \ldots, f_n(x)\}$. The subspace W_n is said to be invariant under the given operator F if $F[W_n] \subseteq W_n$, and then operator F is said to preserve or admit W_n. As in linear algebra, this means

$$F\left[\sum_{i=1}^{n} C_i f_i(x)\right] = \sum_{i=1}^{n} \Psi_i(C_1, \ldots, C_n) f_i(x) \qquad \text{for any} \quad \mathbf{C} = (C_1, \ldots, C_n) \in \mathbb{R}^n,$$

where $\{\Psi_i\}$ are the expansion coefficients of $F[u] \in W_n$ in the basis $\{f_i\}$. It follows that if the linear subspace W_n is invariant with respect to F, then equation (2) has solutions of the form

$$u(x, t) = \sum_{i=1}^{n} C_i(t) f_i(x),$$

where the coefficients $\{C_i(t)\}$ satisfy the n-dimensional dynamical system

$$C_i'(t) = \Psi_i(C_1(t), \ldots, C_n(t)), \qquad i = 1, \ldots, n.$$

Assume that the invariant subspace W_n is defined as the space of solutions of a linear nth-order ODE

$$L[y] \equiv y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0. \tag{3}$$

If the operator $F[u]$ admits the invariant subspace defined by the linear ODE (3), then the invariant condition with respect to F takes the form

$$L[F[u]]|_{[H]} \equiv 0,$$

where $[H]$ denotes the equation $L[u] = 0$ and its differential consequences with respect to x. As was pointed out in [10], the invariant conditions imply that the invariant subspace method is related to the Lie–Bäcklund symmetry and the conditional Lie–Bäcklund symmetry. Some other related approaches were given in [15–25].

To look for the exact solutions of the form

$$\mathbf{U}(x,t) = \mathbf{C}_1(t)f_1(x) + \cdots + \mathbf{C}_n(t)f_n(x) \tag{4}$$

to the system of nonlinear evolution equations

$$\mathbf{U}_t = \mathbf{F}[\mathbf{U}], \tag{5}$$

where $\mathbf{U} = (u^1(x,t), \ldots, u^m(x,t)) \in \mathbb{R}^m$, $\mathbf{C}_1(t), \ldots, \mathbf{C}_m(t) \in \mathbb{R}^m$,

$$\mathbf{F}[\mathbf{U}] \equiv (F^1[\mathbf{U}], \ldots, F^m[\mathbf{U}]) \in \mathbb{R}^m, \tag{6}$$

$F^i[\mathbf{U}] = F^i\left(x, u^1, \ldots, u^m, u_x^1, \ldots, u_x^m, \cdots\right)$ and $F^i(\cdot)$ $(i = 1, \ldots, m)$ are given sufficient smooth functions. Introduce the invariant vector set

$$W_{n,m} = \mathcal{L}\{f_1(x), \ldots, f_n(x)\} \equiv \{\mathbf{C_1}f_1(x) + \cdots + \mathbf{C_n}f_n(x) : \ \mathbf{C_1}, \ldots, \mathbf{C_n} \in \mathbb{R}^m\}$$

as the extension to W_n. $W_{n,m}$ has dimension mn and (4) means that each component belongs to the scalar subspace $W_n = W_{n,1}$.

In paper [1], it was shown by King that the system

$$v_t = (\omega v_x - v\omega_x)_x, \qquad \omega_t = v_{xx} \tag{7}$$

admits the first exact polynomial solution

$$v(x,t) = C_1(t) + C_2(t)x + C_3(t)x^2 + C_4(t)x^3,$$
$$\omega(x,t) = D_1(t) + D_2(t)x + D_3(t)x^2 + D_4(t)x^3,$$

and the second polynomial solution

$$v(x,t) = C_1(t) + C_2(t)x + C_3(t)x^2 + C_4(t)x^3 + C_5(t)x^4,$$
$$\omega(x,t) = D_1(t) + D_2(t)x + D_3(t)x^2.$$

In the second polynomial solution, $C_i(t)$ and $D_i(t)$ satisfy the dynamical system

$$C_1' = 2(D_1C_3 - C_1D_3), \qquad C_2' = 2(D_2C_3 - C_2D_3) + 6D_1C_4,$$
$$C_3' = 6D_2C_4 + 12D_1C_5, \qquad C_4' = 4D_3D_4 + 12D_2C_5, \qquad C_5' = 10D_3C_5,$$
$$D_1' = 2C_3, \qquad D_2' = 6C_4, \qquad D_3' = 12C_5.$$

Indeed, when $f(u,v) = v$, $p(u,v) = -u$, $g(u,v) = 1$ and $r(u,v) = q(u,v) = s(u,v) = 0$, (1) becomes (7), which is a simple model for the solid-state diffusion of a substitutional impurity by a vacancy mechanism. Note that in the second polynomial solution, v and ω belong to different subspaces,

$$v \in W_5 = \mathcal{L}\{1, x, x^2, x^3, x^4\}, \qquad \omega \in W_3 = \mathcal{L}\{1, x, x^2\}.$$

In this paper, we introduce the invariant subspaces $W^1_{n_1} \times \cdots \times W^m_{n_m}$ as a further extension to W_n and a generalization to $W_{n,m}$, where

$$W_{n_i} = \mathcal{L}\{f^i_1(x), \ldots, f^i_{n_i}(x)\}, \qquad i = 1, \ldots, m,$$

and $f^i_1(x), \ldots, f^i_{n_i}(x)$ $(n_i \geqslant 1)$ are linearly independent. If the vector operator $\mathbf{F}$ satisfies the condition

$$\mathbf{F} : W^1_{n_1} \times \cdots \times W^m_{n_m} \longrightarrow W^1_{n_1} \times \cdots \times W^m_{n_m},$$

i.e.

$$F^i : W^1_{n_1} \times \cdots \times W^m_{n_m} \longrightarrow W^i_{n_i}, \qquad i = 1, \ldots, m, \tag{8}$$

then the vector operator $\mathbf{F}$ is said to admit the invariant subspaces $W^1_{n_1} \times \cdots \times W^m_{n_m}$, and exact solutions of system (5) and $\mathbf{U}_{tt} = \mathbf{F}[\mathbf{U}]$ can be constructed in the form

$$u^i = \sum_{j=1}^{n_i} C^i_j(t) f^i_j(x), \qquad i = 1, \ldots, m,$$

with $C^i_j(t)$ satisfying systems of ODEs. In most cases, u^i $(i = 1, \ldots, m)$ belong to different 'scalar' subspaces. Here, we denote by $\mathcal{W}$ the subspace $W^1_{n_1} \times \cdots \times W^m_{n_m}$. The invariant subspace $\mathcal{W}$ has dimension $\sum_{i=1}^m n_i$. Hence the system of evolution equations can be reduced to the $\sum_{i=1}^m n_i$-dimensional dynamical system. Usually, the dynamical system cannot be integrated explicitly, but can be discussed on the phase-plane. It is of interest to extend the scalar operator F to the vector operator $\mathbf{F}$.

Assume that each component $W^i_{n_i} = \mathcal{L}\{f^i_1(x), \ldots, f^i_{n_i}(x)\}$ of the subspace $\mathcal{W}$ is defined as the space generated by the solutions of a linear n_ith-order ODE

$$L^i[y_i] \equiv y_i^{(n_i)} + a^i_{n_i-1}(x) y_i^{(n_i-1)} + \cdots + a^i_1(x) y_i' + a^i_0(x) y_i = 0, \qquad i = 1, \ldots, m. \tag{9}$$

We denote by $[H_i]$ $(i = 1, \ldots, m)$ the equation $L^i[u^i] = 0$ and its differential consequences with respect to x. From (9), the invariance condition of the subspace $\mathcal{W}$ with respect to $\mathbf{F}$ takes the form

$$L^i[F^i[\mathbf{U}]]|_{[H_1]\cap\cdots\cap[H_m]} \equiv 0, \qquad i = 1, \ldots, m. \tag{10}$$

Condition (10) can be interpreted in terms of the Lie–Bäcklund symmetry of system (9) of linear ODEs.

From the invariant condition (10), the maximal dimension of the invariant subspaces preserved by $\mathbf{F}$ can be determined, which is included in the following theorem proved by Svirshchevskii [10] or by Galaktionov and Svirshchevskii [7].

Theorem 1.1. *Assume that a linear subspace W_n is invariant under a nonlinear ordinary differential operator F of order k, then $n \leqslant 2k + 1$.*

The theorem can be naturally extended to the case that $\mathbf{F}$ is a vector. In the case of $m = 2$, when $n_1 = n_2$, the theorem on maximal dimension was given by Galaktionov and Svirshchevskii (see theorem 5.32 in [7]). Without loss of generality, we consider the case $n_1 > n_2$. More generally, we have the following result.

Theorem 1.2. *Let $\mathbf{F}[\mathbf{U}]$ $(m = 2)$ be a nonlinear vector ordinary differential operator of order k. Assume that the system (5) is a really coupled, which means $F_{1,u_2} \neq 0$, $F_{2,u_1} \neq 0$. Without loss of generality, assume that $0 < n_2 \leqslant n_1$. If the nonlinear operator $\mathbf{F}[\mathbf{U}]$ admits the invariant subspace $W^1_{n_1} \times W^2_{n_2}$, then*

$$n_1 - n_2 \leqslant k, \qquad n_2 \leqslant 3k + 1 \tag{11}$$

or

$$n_1 - n_2 \leqslant k, \qquad n_1 \leqslant 4k + 1. \tag{12}$$

Proof. We first prove $n_2 - n_1 \leqslant k$ by a contradiction argument. Assume that

$$n_1 - n_2 > k, \qquad \text{i.e.} \quad n_1 > n_2 + k,$$

and $F[U]$ preserves the invariant subspace $W^1_{n_1} \times W^2_{n_2}$ defined by (9) with $m = 2$. Then the following identities,

$$\begin{aligned} D^{n_1} F^1 &\equiv -\left(a^1_{n_1-1}(x) D^{n_1-1} F^1 + \cdots + a^1_1(x) D F^1 + a^1_0(x) F^1\right), \\ D^{n_2} F^2 &\equiv -\left(a^2_{n_2-1}(x) D^{n_2-1} F^2 + \cdots + a^2_1(x) D F^2 + a^2_0(x) F^2\right), \end{aligned} \tag{13}$$

hold on the solution manifold of (9). Differentiating $F^i\left(x, u^1, u^2, \ldots, u^1_k, u^2_k\right)$ with respect to x and keeping the leading linear and quadratic terms yields

$$\begin{aligned} D F^i &= \sum_{j=1}^{2} u^j_{k+1} \frac{\partial F^i}{\partial u^j_k} + \cdots, \\ D^2 F^i &= \sum_{j=1}^{2} u^j_{k+2} \frac{\partial F^i}{\partial u^j_k} + \sum_{j,l=1}^{2} u^j_{k+1} u^l_{k+1} \frac{\partial^2 F^i}{\partial u^j_k \partial u^l_k} + \cdots, \\ D^3 F^i &= \sum_{j=1}^{2} u^j_{k+3} \frac{\partial F^i}{\partial u^j_k} + 3 \sum_{j,l=1}^{2} u^j_{k+2} u^l_{k+1} \frac{\partial^2 F^i}{\partial u^j_k \partial u^l_k} + \cdots, \\ D^4 F^i &= \sum_{j=1}^{2} u^j_{k+4} \frac{\partial F^i}{\partial u^j_k} + \sum_{j,l=1}^{2} \left[\left(4u^j_{k+3} u^l_{k+1} + 3u^j_{k+2} u^l_{k+2}\right) \frac{\partial^2 F^i}{\partial u^j_k \partial u^l_k}\right] + \cdots, \end{aligned}$$

where

$$u^i_j = \frac{\partial^j u^i}{\partial x^j}, \qquad (i = 1, 2, \ j = 0, 1, \ldots, k, \ u^i_0 = u^i).$$

By induction, for any $p \geqslant 4$, we have

$$\begin{aligned} D^p F^i &= \sum_{j=1}^{2} u^j_{k+p} \frac{\partial F^i}{\partial u^j_k} \\ &\quad + \sum_{j,l=1}^{2} \left[\left(\sum_{s=1}^{[\frac{p}{2}]-1} C^s_p u^j_{k+p-s} u^l_{k+s} + \gamma C^{[\frac{p}{2}]}_p u^j_{k+p-[\frac{p}{2}]} u^l_{k+[\frac{p}{2}]}\right) \frac{\partial^2 F^i}{\partial u^j_k \partial u^l_k}\right] + \cdots, \end{aligned} \tag{14}$$

where $\gamma = 1/2$. In particular, for $p = n_2$ we have

$$\begin{aligned} D^{n_2} F^2 &= \sum_{j=1}^{2} u^j_{k+n_2} \frac{\partial F^2}{\partial u^j_k} \\ &\quad + \sum_{j,l=1}^{2} \left[\left(\sum_{s=1}^{[\frac{n_2}{2}]-1} C^s_{n_2} u^j_{k+n_2-s} u^l_{k+s} + \gamma C^{[\frac{n_2}{2}]}_{n_2} u^j_{k+n_2-[\frac{n_2}{2}]} u^l_{k+[\frac{n_2}{2}]}\right) \frac{\partial^2 F^2}{\partial u^j_k \partial u^l_k}\right] + \cdots. \end{aligned} \tag{15}$$

Note that the term containing the derivative of u^1 with maximal order $k + n_2 (< n_1)$ only appears in $D^{n_2} F^2$ and does not appear in the derivatives $D^p F^2$, $p < n_2$. Taking into account (15) and using (13), equating the coefficients of $u^1_{k+n_2}$ to zero implies that

$$\frac{\partial F^2}{\partial u^1_k} = 0. \tag{16}$$

Similar to the above computation, we find

$$\frac{\partial F^2}{\partial u^1_{k-1}} = 0.$$

By induction, it follows that

$$\frac{\partial F^2}{\partial u^1_j} = 0, \qquad j = 0, 1, \ldots, k,$$

which implies that F^1 does not depend on u^1_j $(j = 0, 1, \ldots, k)$.

Next, we prove $n_2 \leqslant 3k + 1$ or $n_1 \leqslant 4k + 1$ by a contradiction argument. Let us assume that

$$n_2 > 3k + 2 \qquad \text{and} \qquad n_1 > 4k + 2.$$

A direct computation leads to

$$\frac{\partial F^2}{\partial u^j_p \partial u^l_q} = 0, \qquad (j, l = 1, 2,\ p, q = 0, 1, \ldots, k),$$

which implies that F^2 depend on u^j_p linearly, where $j = 1, 2,\ \ p = 0, 1, \ldots, k$. In (14), taking $p = n_1$ leads to

$$D^{n_1} F^1 = \sum_{j=1}^{2} u^j_{k+n_1} \frac{\partial F^1}{\partial u^j_k} + \sum_{j,l=1}^{2} \left[\left(\sum_{s=1}^{[\frac{n_1}{2}]-1} C^s_{n_1} u^j_{k+n_1-s} u^l_{k+s} + \gamma C^{[\frac{n_1}{2}]}_{n_1} u^j_{k+n_1-[\frac{n_1}{2}]} u^l_{k+[\frac{n_1}{2}]} \right) \frac{\partial^2 F^1}{\partial u^j_k \partial u^l_k} \right] + \cdots. \tag{17}$$

The sum in the square brackets in (17) is composed of the quadratic (in higher order derivatives) summands that exhibit the maximal total order of both derivatives:

$$(k + n_1 - s) + k + s = 2k + n_1.$$

Keeping in (17) only the quadratics term, containing at least one derivative of the order not less than $n_1 - 1$ gives

$$D^{n_1} F^1 = \sum_{j,l=1}^{2} \left[\sum_{s=1}^{2k+1} \alpha_s u^j_{k+n_1-s} u^l_{k+s} \frac{\partial^2 F^1}{\partial u^j_k \partial u^l_k} \right] + \cdots, \tag{18}$$

where $\alpha_s = C^s_{n_1},\ \ s = 1, \ldots, 2k$,

$$\alpha_{2k+1} = \begin{cases} C^{2k+1}_{n_1}, & \text{if} \quad n_1 > 4k + 2, \\ \frac{1}{2} C^{2k+1}_{n_1}, & \text{if} \quad n_1 = 4k + 2. \end{cases}$$

Here, we do not display the linear term. By (9), $u^i_{n_i+k}$ for $k \geqslant 0$ can be linearly expressed in terms of $u^i_{n_i-1}, \ldots, u^i_0$, $(i = 1, 2)$. Keeping in the square brackets in (18) the terms containing $u^j_{n_1-(k+1)}$

$$D^{n_1} F^1 = \sum_{j,l=1}^{2} \left[\alpha_{2k+1} u^j_{n_1-(k+1)} u^l_{3k+1} \frac{\partial^2 F^1}{\partial u^j_k \partial u^l_k} \right] + \cdots. \tag{19}$$

Clearly, $n_1 - (k + 1) < n_i,\ \ 3k + 1 < n_i,\ \ (i = 1, 2)$. There exist the following quadratic terms in (19) of the maximal total order $2k + n_1$:

$$\alpha_{2k+1} u^j_{n_1-(k+1)} u^l_{3k+1} \frac{\partial^2 F^1}{\partial u^j_k \partial u^l_k}, \qquad (j, l = 1, 2).$$

It is easy to see that such terms do not appear in the derivatives $D^p F^1$ $(p < n_1)$. In order for (13) to be valid, we should set

$$\frac{\partial^2 F^1}{\partial u_k^j \partial u_k^l} = 0, \qquad (j, l = 1, 2).$$

The following proof is similar to that of theorem 2.8 in [7]. Then we obtain

$$\frac{\partial F^1}{\partial u_p^j \partial u_q^l} = 0, \qquad (j, l = 1, 2, p, q = 0, 1, \ldots, k)$$

which implies that F^1 depends on u_p^j linearly, where $j = 1, 2, \ \ p, q = 0, 1, \ldots, k$. This completes the proof of theorem. □

Here, one should note the following: the particular case of theorem 1.2 was obtained in [7] (see theorem 5.32). The similar arguments can be used in the general case.

The main purpose of this paper is to develop the invariant subspace method for classifying systems of nonlinear diffusion equations (1). The remainder of this paper is organized as follows. In section 2, we determine the functions $f, \ p, \ r, \ g, \ q$ and s in system (1), provided the vector operator (F_1, F_2) admits the invariant subspace $W_{n_1}^1 \times W_{n_2}^2$ defined by ODEs (9) with constant coefficients. The following cases will be considered respectively:

$$\begin{array}{llll}
(n_1, n_2) = (9, 8), & (n_1, n_2) = (9, 7), & (n_1, n_2) = (8, 7), & (n_1, n_2) = (8, 6), \\
(n_1, n_2) = (7, 6), & (n_1, n_2) = (7, 5), & (n_1, n_2) = (6, 5), & (n_1, n_2) = (6, 4), \\
(n_1, n_2) = (5, 5), & (n_1, n_2) = (5, 4), & (n_1, n_2) = (5, 3), & \\
(n_1, n_2) = (4, 4), & (n_1, n_2) = (4, 3), & (n_1, n_2) = (4, 2), & \\
(n_1, n_2) = (3, 3), & (n_1, n_2) = (3, 2), & (n_1, n_2) = (2, 2). &
\end{array}$$

In section 3, to illustrate our approach, we present several examples of exact solutions and reduction of systems with two-component nonlinear diffusion equations to systems of ODEs. For instance, a class of Lokta–Voterra systems with cross and self-diffusion can be reduced to four-dimensional dynamic systems. Behaviour to several solutions of systems (1) is described.

2. Classification of system (1)

In this section, we utilize the invariant condition (10) to perform a classification of systems (1) admitting the invariant subspace $W_{n_1}^1 \times W_{n_2}^2$ defined by ODEs (9). The corresponding solutions called generalized variable separable solutions can be obtained. Assume that the operator $(F_1[u, v], F_2[u, v])$ admits invariant subspaces $W_{n_1}^1 \times W_{n_2}^2$ and each component $W_{n_i}^i$ is defined as the space of the solutions of a linear n_ith-order ODE (9) with constant coefficients, where $n_i \leqslant 5$ $(i = 1, 2)$. The following notation is used throughout the paper:

$$u_j = \frac{\partial^j u}{\partial x^j}, \qquad v_j = \frac{\partial^j v}{\partial x^j} \qquad \text{for all} \quad j \geqslant 0 \ (u_0 = u, \ v_0 = v).$$

2.1. $W_5^1 \times W_5^2$

In this section, we classify all vector operators $(F_1[u, v], F_2[u, v])$ admitting subspaces $W_5^1 \times W_5^2$, defined by the system of linear ODEs

$$\begin{aligned}
L^1[y] &= y^{(5)} + a_4 y^{(4)} + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0, \\
L^2[z] &= z^{(5)} + b_4 z^{(4)} + b_3 z''' + b_2 z'' + b_1 z' + b_0 z = 0.
\end{aligned} \tag{20}$$

In this case, the invariance criterion (10) takes the form

$$G_1 \equiv D^5F_1 + a_4D^4F_1 + a_3D^3F_1 + a_2D^2F_1 + a_1DF_1 + a_0F_1|_{[H_1]\cap[H_2]} = 0, \tag{21}$$

$$G_2 \equiv D^5F_2 + b_4D^4F_2 + b_3D^3F_2 + b_2D^2F_2 + b_1DF_2 + b_0F_2|_{[H_1]\cap[H_2]} = 0. \tag{22}$$

Here and hereafter $[H_1]$ and $[H_2]$ denote respectively to $L^1[u] = 0$ and $L^2[v] = 0$, and their differential consequences with respect to x, and a_i and b_i are constants.

Proposition 2.1. *There are no nonlinear vector operators* $(F_1[u, v], F_2[u, v])$ *preserving invariant subspaces* $W_5^1 \times W_5^2$ *defined by system (20).*

Proof. Note that G_1 and G_2 are the polynomials of u_i and v_i $(i = 1, 2, 3, 4)$. In view of the coefficients of v_3v_4, u_3u_4, v_3u_4 and u_3v_4 in (21), we have

$$35p_v = 0, \qquad 35f_u = 0, \qquad 15p_u + 20f_v = 0, \qquad \text{and} \qquad 15f_v + 20p_u = 0,$$

which imply that $f(u,v)$ and $p(u,v)$ are constants. Plugging them into (21) and collecting coefficients of v_1v_4, u_1u_4 and v_3u_2 yields

$$5r_{vv} = 0, \qquad 5r_{uu} = 0 \qquad \text{and} \qquad 10r_{uv} = 0,$$

which means that $r(u,v)$ is linearly dependent on u and v. Similarly, we deduce that $g(u,v)$ and $q(u,v)$ are also constants, and $s(u,v)$ is linearly dependent on u and v by (22). □

Furthermore, we have the following generalized result.

Proposition 2.1. *There are no nonlinear vector operators as* $(F_1[u, v], F_2[u, v])$ *preserving invariant subspaces* $W_5^1 \times W_5^2$ *defined by systems of the form*

$$L^1[y] = y^{(5)} + a_4(x)y^{(4)} + a_3(x)y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = 0,$$
$$L^2[z] = z^{(5)} + b_4(x)z^{(4)} + b_3(x)z''' + b_2(x)z'' + b_1(x)z' + b_0(x)z = 0.$$

The proof is similar to that for proposition 2.1, we omit the details here.

2.2. $W_{n_1}^1 \times W_{n_2}^2, 6 \leqslant n_1, n_2 \leqslant 9$

In these cases, it was shown in section 1 that system (5) with $m = 2$ admits invariant subspaces defined by (9) when $(n_1, n_2) = (9, 8), (9, 7), (8, 7), (8, 6), (7, 6), (7, 5), (6, 5), (6, 4)$. For system (1), we obtain more precise results.

Proposition 2.2. *There are no nonlinear vector operators* $(F_1[u, v], F_2[u, v])$ *preserving the invariant subspaces* $W_{n_1}^1 \times W_{n_2}^2$ *defined by systems*

$$L^1[y] = y^{(n_1)} + a_{(n_1-1)}y^{(n_1-1)} + \cdots + a_1y' + a_0y = 0,$$
$$L^2[z] = z^{(n_2)} + b_{(n_2-1)}z^{(n_2-1)} + \cdots + b_1z' + b_0z = 0$$

for $(n_1, n_2) = (9, 8), (9, 7), (8, 7), (8, 6), (7, 6), (6, 5)$.

Thus, the possible cases admitting nontrivial invariant subspaces are $(n_1, n_2) = (7, 5)$ and $(n_1, n_2) = (6, 4)$. In fact, we have the following results.

Proposition 2.3. *Assume that the subspace* $W_7^1 \times W_5^2$ *is defined by the system*

$$L^1[y] = y^{(7)} + a_6y^{(6)} + \cdots + a_1y' + a_0y = 0,$$
$$L^2[z] = z^{(5)} + b_4z^{(4)} + \cdots + b_1z' + b_0z = 0. \tag{23}$$

Then nonlinear vector operators $(F_1[u,v], F_2[u,v])$ *possessing the invariant subspaces defined by systems (23) are*

$$\begin{aligned}
F_1[u,v] &= [f_1u_x + (p_1v + p_0)v_x]_x + r_1u + \tfrac{8}{5}p_1b_3v^2 + r_{21}v + r_0,\\
F_2[u,v] &= [g_1v_x + q_1u_x]_x + s_2v + \tfrac{9}{5}b_3q_1u + s_0,\\
L^1[y] &= y^{(7)} + \tfrac{14}{5}b_3y^{(5)} + \tfrac{49}{25}b_3^2y''' + \tfrac{36}{125}b_3^3y' = 0,\\
L_2[z] &= z^{(5)} + b_3z''' + \tfrac{4}{25}b_3^2z' = 0;
\end{aligned}$$

or

$$\begin{aligned}
F_1[u,v] &= [f_1u_x + (p_1v + p_0)v_x]_x + r_1u + \tfrac{9}{10}p_1b_3v^2 + r_{21}v + r_0,\\
F_2[u,v] &= [g_1v_x + q_1u_x]_x + s_2v + \tfrac{16}{5}b_3q_1u + s_0,\\
L^1[y] &= y^{(7)} + \tfrac{21}{5}b_3y^{(5)} + \tfrac{84}{25}b_3^2y''' + \tfrac{64}{125}b_3^3y' = 0,\\
L_2[z] &= z^{(5)} + b_3z''' + \tfrac{4}{25}b_3^2z' = 0.
\end{aligned}$$

Proposition 2.4. *Assume that the subspace* $W_6^1 \times W_4^2$ *is defined by the system*

$$\begin{aligned}
L^1[y] &= y^{(6)} + a_6y^{(5)} + \cdots + a_1y' + a_0y = 0,\\
L^2[z] &= z^{(4)} + b_3z''' + b_2z'' + b_1z' + b_0z = 0.
\end{aligned} \tag{24}$$

Then nonlinear vector operators $(F_1[u,v], F_2[u,v])$ *possessing the invariant subspaces defined by system (24) are given by*

$$\begin{aligned}
F_1[u,v] &= [(-3f_1v + f_0)u_x + (5f_1u + p_2v + p_3)v_x]_x + r_2u + r_1v + r_0,\\
F_2[u,v] &= [g_1v_x + q_1u_x]_x + s_2v + s_3,
\end{aligned}$$

where system (24) is reduced to

$$\begin{aligned}
L^1[y] &= y^{(6)} = 0,\\
L^2[z] &= z^{(4)} = 0.
\end{aligned}$$

2.3. $W_5^1 \times W_4^2$

Assume that the operator $(F_1[u,v], F_2[u,v])$ admits subspaces $W_5^1 \times W_4^2$ defined by the system of linear ODEs

$$\begin{aligned}
L^1[y] &= y^{(5)} + a_4y^{(4)} + a_3y''' + a_2y'' + a_1y' + a_0y = 0,\\
L^2[z] &= z^{(4)} + b_3z''' + b_2z'' + b_1z' + b_0z = 0.
\end{aligned} \tag{25}$$

The invariant condition (10) becomes

$$G_1 \equiv D^5F_1 + a_4D^4F_1 + a_3D^3F_1 + a_2D^2F_1 + a_1DF_1 + a_0F_1|_{[H_1]\cap[H_2]} = 0, \tag{26}$$

$$G_2 \equiv D^4F_2 + b_3D^3F_2 + b_2D^2F_2 + b_1DF_2 + b_0F_2|_{[H_1]\cap[H_2]} = 0. \tag{27}$$

The coefficients of $u_3u_4, v_1u_2u_4, v_2^2, v_3u_4, v_3u_4$ and $v_1v_3u_3$ in G_1 yield the following constraints for the coefficient functions in (1),

$$f_u = 0, \qquad 6f_{uv} + p_{uu} = 0, \qquad p_{vv} = 0, \qquad 4f_v + 3p_u = 0, \qquad 3f_{vv} + 4p_{uv} = 0,$$

which imply that

$$f(u,v) = -3p_1v + f_2, \qquad p(u,v) = 4p_1u + p_2v + p_3. \tag{28}$$

Here and hereafter f_i and p_i are arbitrary constants. Substituting (28) into (26), equating the coefficients of u_2u_3, v_2v_3 and $v_1^2u_3$ in G_1 to zero gives

$$r(u,v) = (r_{11}v + r_{12})u + r_{21}v^2 + r_{22}v + r_{23}. \tag{29}$$

Similarly, from equation (27), we have the following expressions:

$$g(u,v) = g_1, \qquad q(u,v) = q_1, \qquad s(u,v) = s_1u + s_2v + s_3. \tag{30}$$

Here and hereafter r_{ij}, g_i, q_i and s_i also denote arbitrary constants. Plugging (29) and (30) into (26) and (27), respectively, G_1 and G_2 become the polynomials of u_i and v_j $(i = 0,1,2,3,4,\ j = 0,1,2,3)$. Collecting the coefficients, we obtain the following algebraic system for $f_i, p_i, r_{ij}, g_i, q_i, s_i, a_i$ and b_i:

$$\begin{aligned}
(G_1):\ & v_2u_4: a_4p_1 = 0,\\
& v_1u_4: a_3p_1 - 3a_4^2p_1 + 5r_{11} = 0,\\
& v_3u_3: 10a_4p_1 - 35p_1b_3 = 0,\\
& v_2u_3: 19a_3p_1 + 10r_{11} - 35p_1b_2 = 0,\\
& v_1u_3: 4a_4r_{11} + 9a_2p_1 - 3a_4p_1a_3 - 35p_1b_1 = 0,\\
& v_0u_3: p_1b_0 = 0,\\
& v_3u_2: 42p_1b_3^2 - 42p_1b_2 + 12a_3p_1 - 25a_4p_1b_3 + 10r_{11} = 0,\\
& v_2u_2: 6a_4r_{11} + 24a_2p_1 - 25a_4p_1b_2 + 42p_1b_3b_2 - 42p_1b_1 = 0,\\
& v_1u_2: 12a_1p_1 + 3a_3r_{11} + 42p_1b_3b_1 - 42p_1b_0 - 3a_4p_1a_2 - 25a_4p_1b_1 = 0,\\
& v_0u_2: 42p_1b_3b_0 - 25a_4p_1b_0 = 0,\\
& v_3u_1: 4a_4r_{11} + 9a_2p_1 + 17a_4p_1b_3^2 - 17a_4p_1b_2\\
&\qquad - 21p_1b_3^3 + 42p_1b_3b_2 - 13a_3p_1b_3 - 21p_1b_1 - 5r_{11}b_3 = 0,\\
& v_2u_1: -21p_1b_3^2b_2 - 5r_{11}b_2 + 26a_1p_1 - 17a_4p_1b_1 + 3a_3r_{11}\\
&\qquad - 13a_3p_1b_2 + 21p_1b_2^2 + 17a_4p_1b_3b_2 + 21p_1b_3b_1 - 21p_1b_0 = 0,\\
& v_1u_1: 2a_2r_{11} - 17a_4p_1b_0 - 3a_4p_1a_1 + 15a_0p_1 + 21p_1b_2b_1 - 5r_{11}b_1\\
&\qquad - 21p_1b_3^2b_1 + 21p_1b_3b_0 - 13a_3p_1b_1 + 17a_4p_1b_3b_1 = 0,\\
& v_0u_1: 21p_1b_2b_0 - 13a_3p_1b_0 - 21p_1b_3^2b_0 - 5r_{11}b_0 + 17a_4p_1b_3b_0 = 0,\\
& v_3u_0: a_3r_{11} - 4a_3p_1b_2 + 4p_1b_3^4 + 8a_4p_1b_3b_2 + 4p_1b_2^2 - 12p_1b_3^2b_2\\
&\qquad - a_4r_{11}b_3 + r_{11}b_3^2 - r_{11}b_2 - 4p_1b_0 - 4a_2p_1b_3 + 4a_3p_1b_3^2\\
&\qquad + 4a_1p_1 - 4a_4p_1b_1 - 4a_4p_1b_3^3 + 8p_1b_3b_1 = 0,\\
& v_2u_0: 4p_1b_3^3b_2 - r_{11}b_1 - 4a_2p_1b_2 - 4a_4p_1b_0 + 4a_4p_1b_2^2 - 4p_1b_3^2b_1\\
&\qquad + r_{11}b_3b_2 + 4p_1b_3b_0 - 8p_1b_3b_2^2 - 4a_4p_1b_3^2b_2 + 4a_4p_1b_3b_1 + 4a_3p_1b_3b_2\\
&\qquad - a_4r_{11}b_2 + 8p_1b_2b_1 - 4a_3p_1b_1 + a_2r_{11} + 25a_0p_1 = 0,
\end{aligned} \tag{31}$$

$$
\begin{aligned}
v_1u_0 :\ & 4p_1b_1^2 - 4p_1b_3^2b_0 - 3a_4p_1a_0 - 4a_3p_1b_0 - a_4r_{11}b_1 + r_{11}b_3b_1 \\
& - 4a_2p_1b_1 - 4a_4p_1b_3^2b_1 + a_1r_{11} + 4a_4p_1b_2b_1 - r_{11}b_0 + 4a_3p_1b_3b_1 \\
& + 4a_4p_1b_3b_0 + 4p_1b_3^3b_1 - 8p_1b_3b_2b_1 + 4p_1b_2b_0 = 0, \\
v_0u_0 :\ & 4p_1b_1b_0 + 4a_3p_1b_3b_0 + r_{11}b_3b_0 - 8p_1b_2b_3b_0 - 4a_2p_1b_0 \\
& + 4a_4p_1b_2b_0 - a_4r_{11}b_0 - 4a_4p_1b_3^2b_0 + 4p_1b_3^3b_0 = 0, \\
v_3^2 :\ & 10a_4p_2 - 35p_2b_3 = 0, \\
v_2v_3 :\ & 10a_3p_2 + 20r_{21} + 21p_2b_3^2 - 15a_4p_2b_3 - 56p_2b_2 = 0, \\
v_1v_3 :\ & 4a_2p_2 + 14p_2b_3b_2 + 6a_4p_2b_3^2 - 5a_3p_2b_3 - 42p_2b_1 - 6a_4p_2b_2 \\
& - 7p_2b_3^3 + 8a_4r_{21} - 10r_{21}b_3 = 0, \\
v_0v_3 :\ & 2a_3r_{21} - 36p_2b_0 + p_2b_2^2 + a_3p_2b_3^2 - a_3p_2b_2 + p_2b_3^4 - a_2p_2b_3 \\
& 3p_2b_3^2b_2 + 2p_2b_3b_1 - 2a_4r_{21}b_3 + 2a_4p_2b_3b_2 + a_1p_2 + 2r_{21}b_3^2 \\
& - 2r_{21}b_2 - a_4p_2b_1 - a_4p_2b_3^3 = 0, \\
v_3 :\ & r_{22}b_3^2 - p_3b_0 - 3p_3b_3^2b_2 + 2p_3b_3b_1 - a_2p_3b_3 - a_4r_{22}b_3 \\
& + p_3b_3^4 - a_4p_3b_1 - a_4p_3b_3^3 + a_3p_3b_3^2 - a_3p_3b_2 - r_{22}b_2 \\
& + p_3b_2^2 + 2a_4p_3b_3b_2 + a_3r_{22} + a_1p_3 = 0, \\
v_2^2 :\ & 21p_2b_3b_2 - 15a_4p_2b_2 + 3a_2p_2 - 21p_2b_1 + 6a_4r_{21} = 0, \\
v_1v_2 :\ & 7p_2b_2^2 + 3a_1p_2 - 21a_4p_2b_1 + 28p_2b_3b_1 - 10r_{21}b_2 - 5a_3p_2b_2 \\
& - 7p_2b_3^2b_2 - 28p_2b_0 + 6a_4p_2b_3b_2 + 6a_3r_{21} = 0, \\
v_0v_2 :\ & a_4p_2b_2^2 - 16a_4p_2b_0 - a_3p_2b_1 + a_0p_2 - a_2p_2b_2 + p_2b_3^3b_2 \\
& - p_2b_3^2b_1 - 2a_4r_{21}b_2 - 2p_2b_3b_2^2 + 2p_2b_2b_1 - 2r_{21}b_1 + 22p_2b_3b_0 \\
& + 2a_2r_{21} + 2r_{21}b_3b_2 - a_4p_2b_3^2b_2 + a_4p_2b_3b_1 + a_3p_2b_3b_2 = 0, \\
v_2 :\ & a_0p_3 - r_{22}b_1 + a_2r_{22} - a_4r_{22}b_2 - 2p_3b_3b_2^2 + p_3b_3^3b_2 - p_3b_3^2b_1 \\
& + p_3b_3b_0 + a_4p_3b_2^2 + 2p_3b_2b_1 + r_{22}b_3b_2 - a_4p_3b_0 + a_3p_3b_3b_2 - a_3p_3b_1 \\
& - a_4p_3b_3^2b_2 - a_2p_3b_2 + a_4p_3b_3b_1 = 0, \\
v_1^2 :\ & 7p_2b_3b_0 - 10r_{21}b_1 + a_0p_2 - 5a_3p_2b_1 - 7p_2b_3^2b_1 + 6a_4p_2b_3b_1 + 7p_2b_2b_1 \\
& - 6a_4p_2b_0 + 2a_2r_{21} = 0, \\
v_0v_1 :\ & 2r_{21}b_1b_3 - a_2p_2b_1 - 2a_4r_{21}b_1 - a_4p_2b_3^2b_1 + 7a_4p_2b_3b_0 \\
& - 8p_2b_3^2b_0 + a_3p_2b_3b_1 + 8p_2b_2b_0 + p_2b_1^2 + p_2b_3^3b_1 - 2p_2b_3b_2b_1 + 2a_1r_{21} \\
& - 12r_{21}b_0 + a_4p_2b_2b_1 - 6a_3p_2b_0 = 0, \\
v_1 :\ & a_3p_3b_3b_1 + p_3b_3^3b_1 - p_3b_3^2b_0 + p_3b_1^2 + a_1r_{22} + r_{22}b_3b_1 - 2p_3b_3b_2b_1 + p_3b_2b_0 \\
& + a_4p_3b_2b_1 - a_3p_3b_0 - a_4p_3b_3^2b_1 + a_4p_3b_3b_0 - a_4r_{22}b_1 - r_{22}b_0 - a_2p_3b_1 = 0, \\
v_0^2 :\ & 2r_{21}b_3b_0 - 2p_2b_2b_3b_0 - a_4p_2b_3^2b_0 + a_4p_2b_2b_0 + a_0r_{21} - 2a_4r_{21}b_0 \\
& + p_2b_1b_0 - a_2p_2b_0 + a_3p_2b_3b_0 + p_2b_3^3b_0 = 0, \\
v_0 :\ & p_3b_3^3b_0 - a_4p_3b_3^2b_0 + a_4p_3b_2b_0 + a_0r_{22} - a_2p_3b_0 + p_3b_1b_0 - a_4r_{22}b_0 \\
& + r_{22}b_3b_0 - 2p_3b_2b_3b_0 + a_3p_3b_3b_0 = 0, \\
v_0^0 :\ & a_0r_{23} = 0,
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
(G_2)\quad u_4 :\ & q_1a_4^2 + s_1 + b_2q_1 - q_1a_3 - b_3qa_4 = 0, \\
u_3 :\ & q_1a_4a_3 - b_3q_1a_3 - q_1a_2 + b_3s_1 + b_1q_1 = 0, \\
u_2 :\ & b_2s_1 - q_1a_1 + b_0q_1 - b_3q_1a_2 + q_1a_4a_2 = 0, \\
u_1 :\ & q_1a_4a_1 - q_1a_0 - b_3q_1a_1 + b_1s_1 = 0, \\
u_0 :\ & q_1a_4a_0 + b_0s_1 - b_3q_1a_0 = 0, \\
1 :\ & b_0s_3 = 0.
\end{aligned}
\tag{33}
$$

Note that $q_1 \neq 0$ and $p_1 \neq 0$. Solving this system, we obtain the following result.

Table 1(I). Operators (35) preserving invariant subspace $W_5^3 \times W_3^2$ defined by system (34).

Number	Constant coefficients in (35)	System (34)
1	$p_1 = -\frac{5}{2}f_1,\quad p_{21} = 0,\quad r_{11} = -3b_1 f_1,\quad r_{21} = 0,$ $r_{22} = 2b_1 p_{22},\quad s_1 = \frac{1}{4}b_1 q_1,\quad s_{21} = 2b_1 g_1$	$L^1[y] = y^{(5)} + \frac{5}{4}b_1 y''' + \frac{1}{4}b_1^2 y' = 0,$ $L^2[z] = z''' + b_1 z' = 0$
2	$r_{11} = 3(2f_1 + p_1),\quad r_{21} = 3b_1 p_{21},\quad s_1 = 4b_1 q_1,$ $s_{21} = 2b_1 g_1$	$L^1[y] = y^{(5)} + 5b_1 y''' + 4b_1^2 y' = 0,$ $L^2[z] = z''' + b_1 z' = 0$
3	$f_1 = p_1 = r_{11} = 0,\quad r_{21} = \frac{4}{3}b_1 p_{21},$ $r_{22} = 2b_1 p_{22},\quad s_1 = 9b_1 q_1,\quad s_{21} = 2b_1 g_1$	$L^1[y] = y^{(5)} + 10b_1 y''' + 9b_1^2 y' = 0,$ $L^2[z] = z''' + b_1 z' = 0$
4	$f_1 = p_1 = p_{21} = r_{11} = r_{21} = 0,\quad r_{22} = 2b_1 p_{22},$ $g_1 = 0,\quad s_1 = (a_3 - b_1)q_1,\quad s_{21} = 0$	$L^1[y] = y^{(5)} + a_3 y''' + b_1(a_3 - b_1)y' = 0,$ $L^2[z] = z''' + b_1 z' = 0$
5	$r_{11} = r_{21} = s_1 = s_{21} = 0$	$L^1[y] = y^{(5)} = 0,\quad L^2[z] = z''' = 0$

Proposition 2.5. *There exists a nonlinear vector operator* $(F_1[u, v], F_2[u, v])$ *admitting the subspace* $W_5^1 \times W_4^2$ *defined by system (25), which is*

$$\begin{aligned} F_1[u, v] &= [(-3p_1 v + f_2)u_x + (4p_1 u + p_2 v + p_3)v_x]_x + r_{12}u + r_{21}v + r_{22},\\ F_2[u, v] &= [g_1 v_x + q_1 u_x]_x + s_2 v + s_3, \end{aligned}$$

with $W_5^1 = \{1, x, x^2, x^3, x^4\}$ *and* $W_4^2 = \{1, x, x^2, x^3\}(L^1[y] = y^{(5)} = 0,$ *and* $L^2[z] = z^{(4)} = 0)$

2.4. $W_5^1 \times W_3^2$

In this section, we give a description of all vector operators $(F_1[u, v], F_2[u, v])$ admitting subspaces $W_5^1 \times W_3^2$ defined by the system of linear ODEs

$$\begin{aligned} L^1[y] &= y^{(5)} + a_4 y^{(4)} + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0,\\ L^2[z] &= z''' + b_2 z'' + b_1 z' + b_0 z = 0. \end{aligned} \tag{34}$$

The similar calculation as that in section 2.3 leads to the following result.

Proposition 2.6. *Any nonlinear vector operator* $(F_1[u, v], F_2[u, v])$ *possessing the invariant subspace defined by system (34) takes the form*

$$\begin{aligned} F_1[u, v] &= [(f_1 v + f_2)u_x + (p_1 u + p_{21}v^2 + p_{22}v + p_{23})v_x]_x\\ &\quad + (r_{11}v + r_{12})u + r_{21}v^3 + r_{22}v^2 + r_{23}v + r_{24},\\ F_2[u, v] &= [(g_1 v + g_2)v_x + q_1 u_x]_x + s_1 u + s_{21}v^2 + s_{22}v + s_{23}. \end{aligned} \tag{35}$$

The full description of the operators (35) and the corresponding systems (34) are listed in table 1(I).

Solving the systems as (34) in table 1(I) yields the corresponding invariant subspaces. Here, we give the expressions of the invariant subspace in the fourth case. Similarly, it is easy to obtain the invariant subspaces of the other four cases. Clearly, in the fourth case,

$$W_3^2 = \begin{cases} \mathcal{L}\{1, \cos(\sqrt{b_1}x), \sin(\sqrt{b_1}x)\} & \text{for} \quad b_1 > 0,\\ \mathcal{L}\{1, \exp(-\sqrt{-b_1}x), \exp(\sqrt{-b_1}x)\} & \text{for} \quad b_1 < 0,\\ \mathcal{L}\{1, x, x^2\} & \text{for} \quad b_1 = 0, \end{cases}$$

and expressions for W_5^1 are listed in table 1(II).

Table 1(II). Expression for W_5^1 in the fourth case of table 1(I).

a_3-2b_1	a_3-b_1	b_1	W_5^1
$a_3=2b_1$	$a_3-b_1=b_1$	$b_1>0$	$\mathcal{L}\{1, \cos(\sqrt{b_1}x), \sin(\sqrt{b_1}x), x\cos(\sqrt{b_1}x), x\sin(\sqrt{b_1}x)\}$
		$b_1<0$	$\mathcal{L}\{1, \exp(-\sqrt{-b_1}x), \exp(\sqrt{-b_1}x), x\exp(-\sqrt{-b_1}x), x\exp(\sqrt{-b_1}x)\}$
	$a_3-b_1>0$	$b_1>0$	$\mathcal{L}\{1, \cos(\sqrt{b_1}x), \sin(\sqrt{b_1}x), \cos(\sqrt{a_3-b_1}x), \sin(\sqrt{a_3-b_1}x)\}$
		$b_1=0$	$\mathcal{L}\{1, x, x^2, \cos(\sqrt{a_3}x), \sin(\sqrt{a_3}x)\}$
		$b_1<0$	$\mathcal{L}\{1, \exp(-\sqrt{-b_1}x), \exp(-\sqrt{b_1}x), \cos(\sqrt{a_3-b_1}x), \sin(\sqrt{a_3-b_1}x)\}$
$a_3\neq 2b_1$	$a_3-b_1=0$	$b_1>0$	$\mathcal{L}\{1, \cos(\sqrt{b_1}x), \sin(\sqrt{b_1}x), x, x^2\}$
		$b_1<0$	$\mathcal{L}\{1, \exp(-\sqrt{-b_1}x), \exp(\sqrt{-b_1}x), x, x^2\}$
	$a_3-b_1<0$	$b_1>0$	$\mathcal{L}\{1, \cos(\sqrt{b_1}x), \sin(\sqrt{b_1}x), \exp(-\sqrt{b_1-a_3}x), \sin(\sqrt{b_1-a_3}x)\}$
		$b_1=0$	$\mathcal{L}\{1, x, x^2, \exp(-\sqrt{-a_3}x), \exp(\sqrt{-a_3}x)\}$
		$b_1<0$	$\mathcal{L}\{1, \exp(-\sqrt{-b_1}x), \sin(\sqrt{b_1}x), \exp(-\sqrt{b_1-a_3}x), \exp(\sqrt{b_1-a_3}x)\}$

2.5. $W_4^1 \times W_4^2$

In this section, we give the description of all vector operators $(F_1[u, v], F_2[u, v])$ admitting subspaces $W_4^1 \times W_4^2$ defined by the system of linear ODEs

$$
\begin{aligned}
L^1[y] &= y^{(4)} + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0,\\
L^2[z] &= z^{(4)} + b_3 z''' + b_2 z'' + b_1 z' + b_0 z = 0.
\end{aligned} \tag{36}
$$

From the invariant condition, the similar calculation as above gives the following result.

Proposition 2.7. *There exist nonlinear vector operators $(F_1[u, v], F_2[u, v])$ admitting the subspace $W_4^1 \times W_4^2$ defined by the system (36), which are*

$$
\begin{aligned}
F_1[u, v] &= [(-p_1 v + f_2)u_x + (p_1 u + p_2)v_x]_x + r_{12}u + r_{21}v + r_{22},\\
F_2[u, v] &= [(-q_1 v + g_2)v_x + (q_1 v + q_2)u_x]_x + s_{12}u + s_{21}v + s_{22},
\end{aligned}
$$

with $W_4^1 = W_4^2 = \mathcal{L}\{1, x, x^2, x^3\}$.

2.6. $W_4^1 \times W_3^2$

In this section, we give a description of all vector operators $(F_1[u, v], F_2[u, v])$ admitting subspaces $W_4^1 \times W_3^2$ defined by the system of linear ODEs

$$
\begin{aligned}
L^1[y] &= y^{(4)} + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0,\\
L^2[z] &= z''' + b_2 z'' + b_1 z' + b_0 z = 0.
\end{aligned} \tag{37}
$$

Similarly, the following result is deduced from the invariant condition.

Proposition 2.8. *Any nonlinear vector operator $(F_1[u, v], F_2[u, v])$ possessing the invariant subspace defined by system (37) takes the form*

$$
\begin{aligned}
F_1[u, v] &= [(f_1 v + f_2)u_x + (p_1 u + p_2 v + p_3)v_x]_x + (r_{11}v + r_{12})u + r_{21}v^2 + r_{22}v + r_{23},\\
F_2[u, v] &= \left[\left(-\tfrac{3}{2}q_1 u + g_{21}v + g_{22}\right)v_x + (q_1 v + q_2)u_x\right]_x + s_{12}u + s_{21}v^2 + s_{22}v + s_{23}.
\end{aligned} \tag{38}
$$

The full description of the operators (38) and the corresponding systems (37) are listed in table 2.

Table 2. Operators (38) preserving invariant subspace $W_4^1 \times W_3^2$ defined by system (37).

Number	Constant coefficients in (38)	System (37)
1	$p_1 = -2f_1,\ \ r_{11} = -2b_1 f_1,\ \ r_{21} = 2b_1 p_2,$ $q_1 = 0,\ \ s_{12} = 0,\ \ s_{21} = 2b_1 g_{21}$	$L^1[y] = y^{(4)} + b_1 y'' = 0,$ $L^2[z] = z''' + b_1 z' = 0,$
2	$p_1 = -4f_1,\ \ r_{11} = \frac{3}{2}a_3^2 f_1,\ \ r_{21} = -\frac{1}{2}a_3^2 p_2,$ $q_1 = 0,\ \ s_{12} = -a_3^2 q_2,\ \ s_{21} = -\frac{1}{2}a_3^2 g_{21}$	$L^1[y] = y^{(4)} + a_3 y''' - \frac{1}{4}a_3^2 y'' - \frac{1}{4}a_3^3 y' = 0,$ $L^2[z] = z''' - \frac{1}{4}a_3^2 z' = 0$
3	$r_{11} = r_{21} = s_{12} = s_{21} = 0$	$L^1[y] = y^{(4)} = 0,\ \ L^2[z] = z''' = 0$

2.7. $W_4^1 \times W_2^2$

In this section, we present all vector operators $(F_1[u, v], F_2[u, v])$ admitting the subspaces $W_4^1 \times W_2^2$ given by the systems of linear ODEs

$$\begin{aligned} L^1[y] &= y^{(4)} + a_3 y''' + a_2 y'' + a_1 y' + a_0 y = 0,\\ L^2[z] &= z'' + b_1 z' + b_0 z = 0. \end{aligned} \tag{39}$$

In this case, we have the following result from the invariant condition.

Proposition 2.9. *Any nonlinear vector operator* $(F_1[u, v], F_2[u, v])$ *preserving the invariant subspace* $W_4^1 \times W_2^2$ *defined by system (39) takes the form*

$$\begin{aligned} F_1[u, v] &= \big\{\big(-\tfrac{1}{3}p_{13}v^3 + f_1 v^2 + f_2 v + f_3\big)u_x\\ &\quad + [(p_{13}v^2 + p_{14}v + p_{15})u + p_{21}v^4 + p_{22}v^3 + p_{23}v^2 + p_{24}v + p_{25}]v_x\big\}_x\\ &\quad + (r_{12}v^2 + r_{13}v + r_{14})u + r_{21}v^5 + r_{22}v^4 + r_{23}v^3 + r_{24}v^2 + r_{25}v + r_{26},\\ F_2[u, v] &= \big[(g_{13}u + g_{21}v^2 + g_{22}v + g_{23})v_x + \big(-\tfrac{1}{3}g_{13}v + q_1\big)u_x\big]_x\\ &\quad + (s_{11}v + s_{12})u + s_{21}v^3 + s_{22}v^2 + s_{23}v + s_{24}. \end{aligned} \tag{40}$$

The full description of the operators (40) and the corresponding systems (39) are listed in table 3.

2.8. $W_3^1 \times W_3^2$

We describe all vector operators $(F_1[u, v], F_2[u, v])$ preserving subspaces $W_3^1 \times W_3^2$ defined by the systems of linear ODEs

$$\begin{aligned} L^1[y] &= y''' + a_2 y'' + a_1 y' + a_0 y = 0,\\ L^2[z] &= z''' + b_2 z'' + b_1 z' + b_0 z = 0. \end{aligned} \tag{41}$$

In this case, we have the following result from the invariant condition.

Proposition 2.10. *Any nonlinear vector operator* $(F_1[u, v], F_2[u, v])$ *preserving the invariant subspace* $W_3^1 \times W_3^2$ *defined by system (41) takes the form*

$$\begin{aligned} F_1[u, v] &= [(f_1 u + f_{21}v + f_{22})u_x + (p_1 v + p_{21}u + p_{22})v_x]_x\\ &\quad + r_1 u^2 + (r_{21}v + r_{22})u + r_{31}v^2 + r_{32}v + r_{33},\\ F_2[u, v] &= [(g_1 v + g_{21}u + g_{22})v_x + (q_1 u + q_{21}v + q_{22})u_x]_x\\ &\quad + s_1 u^2 + (s_{21}v + s_{22})u + s_{31}v^2 + s_{32}v + s_{33}. \end{aligned} \tag{42}$$

The full description of the operators (42) and the corresponding systems (41) are presented in table 4.

Table 3. Operators (40) preserving invariant subspace $W_4^1 \times W_2^2$ defined by system (39).

Parameters in (39)	Number	Operators (40)	System (39)
$b_1 = b_0 = 0$	1	$F_1 = [f_3 u_x + (p_{15} u + p_{23} v^2 + p_{24} v + p_{25}) v_x]_x + r_{14} u + r_{25} v + r_{26},$ $F_2 = [(g_{21} v^2 + g_{22} v + g_{23}) v_x + q_1 u_x]_x + a_2 q_1 u + s_{23} v + s_{24}$	$L^1[y] = y^{(4)} + a_2 y'' = 0,$ $L^2[z] = z'' = 0$
	2	$F_1 = \{(-\frac{1}{3} p_{13} v^3 + f_1 v^2 + f_2 v + f_3) u_x + [(p_{13} v^2 + p_{14} v + p_{15}) u$ $+ p_{21} v^4 + p_{22} v^3 + p_{23} v^2 + p_{24} v + p_{25}] v_x\}_x + r_{14} u + r_{23} v^3 + r_{24} v^2 + r_{25} v + r_{26},$ $F_2 = [(g_{13} u + g_{21} v^2 + g_{22} v + g_{23}) v_x + (-\frac{1}{3} g_{13} v + q_1) u_x]_x + s_{23} v + s_{24}$	$L^1[y] = y^{(4)} = 0,$ $L^2[z] = z'' = 0$
$b_1 = 0,\ b_0 \neq 0$	1	$F_1 = [f_3 u_x + (p_{23} v^2 + p_{25}) v_x]_x + r_{14} u + 3 b_0 p_{23} v^3 + r_{25} v,$ $F_2 = [(g_{21} v^2 + g_{23}) v_x + q_1 u_x]_x + a_0 q_1 u + 3 b_0 g_{21} v^3 + s_{23} v$	$L^1[y] = y^{(4)} + (a_0 + b_0) y'' + a_0 b_0 y = 0,$ $L^2[z] = z'' + b_0 z = 0$
	2	$F_1 = [(f_2 v + f_3) u_x + (-2 f_2 u + p_{23} v^2 + p_{24} v + p_{25}) v_x]_x + r_{14} u + \frac{3}{4} b_0 p_{23} v^3 + r_{25} v,$ $F_2 = [(g_{21} v^2 + g_{23}) v_x + q_1 u_x]_x + b_0 q_1 u + \frac{3}{4} b_0 g_{21} v^3 + s_{23} v$	$L^1[y] = y^{(4)} + \frac{5}{4} b_0 y'' + \frac{1}{4} b_0^2 y = 0,$ $L^2[z] = z'' + b_0 z = 0$
	3	$F_1 = [(f_1 v^2 + f_3) u_x + (-4 f_1 u v + p_{23} v^2 + p_{25}) v_x]_x + (-9 f_1 b_0 v^2 + r_{14}) u + 3 p_{23} b_0 v^3 + r_{25} v,$ $F_2 = [(g_{21} v^2 + g_{23}) v_x + q_1 u_x]_x + b_0 q_1 u + 3 b_0 g_{21} v^3 + s_{23} v$	$L^1[y] = y^{(4)} + 2 b_0 y'' + b_0^2 y = 0,$ $L^2[z] = z'' + b_0 z = 0$
	4	$F_1 = [(f_1 v^2 + f_3) u_x + (p_{14} u v + p_{21} v^4 + p_{23} v^2 + p_{25}) v_x]_x$ $+ [5 b_0 (3 f_1 + p_{14}) v^2 + r_{14}] u + 5 b_0 p_{21} v^5 + r_{23} v^3 + r_{25} v,$ $F_2 = [(g_{21} v^2 + g_{23}) v_x + q_1 u_x]_x + 9 b_0 q_1 u + 3 b_0 g_{21} v^3 + s_{23} v$	$L^1[y] = y^{(4)} + 10 b_0 y'' + 9 b_0^2 y = 0,$ $L^2[z] = z'' + b_0 z = 0$
	5	$F_1 = [f_3 u_x + (p_{21} v^4 + p_{23} v^2 + p_{25}) v_x]_x + r_{14} u + \frac{9}{5} b_0 p_{21} v^5 + 3 b_0 p_{23} v^3 + r_{25} v,$ $F_2 = [(g_{21} v^2 + g_{23}) v_x + q_1 u_x]_x + 25 b_0 q_1 u + 3 b_0 g_{21} v^3 + s_{23} v$	$L^1[y] = y^{(4)} + 26 b_0 y'' + 25 b_0^2 y = 0,$ $L^2[z] = z'' + b_0 z = 0$
$b_1 \neq 0,\ b_0 = 0$	1	$F_1 = [(f_2 v + f_3) u_x + (-f_2 u + p_{23} v^2 + p_{24} v + p_{25}) v_x]_x$ $+ (-3 f_2 b_1^2 v + r_{14}) u - 3 p_{23} b_1^2 v^3 + r_{24} v^2 + r_{25} v + r_{26},$ $F_2 = [(g_{22} v + g_{23}) v_x + q_1 u_x]_x - 4 b_1^2 q_1 u - 2 g_{22} b_1^2 v^2 + s_{33} v + s_{24}$	$L^1[y] = y^{(4)} + b_1 y''' - 4 b_1^2 y'' - 4 b_1^3 y' = 0,$ $L^2[z] = z'' + b_1 z' = 0$
	2	$F_1 = [(f_2 v + f_3) u_x + (-\frac{5}{2} f_2 u + p_{24} v + p_{25}) v_x]_x + (3 f_2 b_1^2 v + r_{14}) u - 2 b_1^2 p_{24} v^2 + r_{25} v + r_{26},$ $F_2 = [(g_{22} v + g_{23}) v_x + q_1 u_x]_x - \frac{1}{4} b_1^2 q_1 u - 2 g_{22} b_1^2 v^2 + s_{33} v + s_{24}$	$L^1[y] = y^{(4)} + b_1 y''' - \frac{1}{4} b_1^2 y'' - \frac{1}{4} b_1^3 y' = 0,$ $L^2[z] = z'' + b_1 z' = 0$
	3	$F_1 = [(f_2 v + f_3) u_x + (-3 f_2 u + p_{24} v + p_{25}) v_x]_x + (4 f_2 b_1^2 v + r_{14}) u - 2 b_1^2 p_{24} v^2 + r_{25} v + r_{26},$ $F_2 = [(g_{22} v + g_{23}) v_x + q_1 u_x]_x - b_1^2 q_1 u - 2 g_{22} b_1^2 v^2 + s_{33} v + s_{24}$	$L^1[y] = y^{(4)} + b_1 y''' - b_1^2 y'' - b_1^3 y' = 0,$ $L^2[z] = z'' + b_1 z' = 0$
	4	$F_1 = [f_3 u_x + (p_{23} v^2 + p_{24} v + p_{25}) v_x]_x + r_{14} u - \frac{4}{3} p_{23} b_1^2 v^3 - 2 b_1^2 p_{24} v^2 + r_{25} v + r_{26},$ $F_2 = [(g_{22} v + g_{23}) v_x + q_1 u_x]_x - 9 b_1^2 q_1 u - 2 g_{22} b_1^2 v^2 + s_{33} v + s_{24}$	$L^1[y] = y^{(4)} + b_1 y''' - 9 b_1^2 y'' - 9 b_1^3 y' = 0,$ $L^2[z] = z'' + b_1 z' = 0$
	5	$F_1 = [f_3 u_x + (p_{24} v + p_{25}) v_x]_x + r_{14} u - 2 b_1^2 p_{24} v^2 + r_{25} v + r_{26},$ $F_2 = [(g_{22} v + g_{23}) v_x + q_1 u_x]_x + a_2 q_1 u - 2 g_{22} b_1^2 v^2 + s_{33} v + s_{24}$	$L^1[y] = y^{(4)} + b_1 y''' + a_2 y'' + a_2 b_1 y' = 0,$ $L^2[z] = z'' + b_1 z' = 0$

Remark. There are no operators (40) preserving invariant subspaces $W_4^1 \times W_2^2$ defined by systems (39) for $b_1 \neq 0$ and $b_0 \neq 0$.

Table 4. Operators (42) admitting invariant subspace $W_3^1 \times W_3^2$ defined by system (41).

Number	Operators (42)	System (41)
1	$F_1 = [(f_1 u + f_{22})u_x + p_{22}v_x]_x - 2b_2^2 f_1 u^2 + r_{22}u + r_{33},$ $F_2 = [g_{22}v_x + q_{22}u_x]_x - b_2^2 q_{22}u + s_{32}v + s_{33}$	$L^1[y] = y''' - b_2^2 y' = 0,$ $L^2[z] = z''' + b_2 z'' = 0$
2	$F_1 = [(f_1 u + f_{22})u_x + p_{22}v_x]_x - \frac{2}{9}b_2^2 f_1 u^2 + r_{22}u - \frac{4}{9}p_{22}b_2^2 v + r_{33},$ $F_2 = [(g_{21}u + g_{22})v_x + (-3g_{21}v + q_{22})u_x]_x + b_2^2(\frac{1}{3}g_{21}v - \frac{1}{9}q_{22})u + s_{32}v + s_{33}$	$L_1[y] = y''' - \frac{1}{9}b_2^2 y' = 0,$ $L_2[z] = z''' + b_2 z'' + \frac{2}{9}b_2^2 z' = 0$
3	$F_1 = [(f_1 u + f_{21}v + f_{22})u_x + (-\frac{3}{2}f_{21}u + p_{22})v_x]_x + \frac{1}{2}b_1 f_1 u^2 + (-\frac{3}{2}f_{21}b_1 v + r_{22})u + b_1 p_{22}v + r_{33},$ $F_2 = [(g_1 v + g_{22})v_x + (q_1 u + 4q_{22})u_x]_x + \frac{1}{8}b_1 q_1 u^2 + b_1 q_{22}u + 2b_1 g_1 v^2 + s_{32}v + s_{33}$	$L_1[y] = y''' + \frac{1}{4}b_1 y' = 0,$ $L_2[z] = z''' + b_1 z' = 0$
4	$F_1 = [(f_1 u + f_{22})u_x + p_{22}v_x]_x + 2a_1 f_1 u^2 + r_{22}u + p_{22}b_1 v + r_{33},$ $F_2 = [(g_1 v + g_{22})v_x + q_{22}u_x]_x + q_{22}a_1 u + 2b_1 g_1 v^2 + s_{32}v + s_{33}$	$L_1[y] = y''' + a_1 y' = 0,$ $L_2[z] = z''' + b_1 z' = 0$
5	$F_1 = [(f_1 u + f_{22})u_x + p_{22}v_x]_x + 2a_1 f_1 u^2 + r_{22}u - p_{22}b_2^2 v + r_{33},$ $F_2 = [g_{22}v_x + q_{22}u_x]_x + s_{32}v$	$L_1[y] = y''' + a_1 y' = 0,$ $L_2[z] = z''' + b_2 z'' + a_1 z' + a_1 b_2 z = 0$
6	$F_1 = [(f_1 u + f_{21}v + f_{22})u_x + (p_1 v + p_{21}u + p_{22})v_x]_x$ $+ 2b_1 f_1 u^2 + [2b_1(f_{21} + p_{21})v + r_{22}]u + 2p_1 b_1 v^2 + r_{32}v + r_{33},$ $F_2 = [(g_1 v + g_{21}u + g_{22})v_x + (q_1 u + q_{21}v + q_{22})u_x]_x$ $+ 2b_1 q_1 u^2 + [2b_1(q_{21} + g_{21})v + s_{22}]u + 2b_1 g_1 v^2 + s_{32}v + s_{33}$	$L^1[y] = y''' + b_1 y' = 0,$ $L^2[z] = z''' + b_1 z' = 0$

2.9. $W_3^1 \times W_2^2$

In this section, we present the classification of all vector operators $(F_1[u, v], F_2[u, v])$ admitting the invariant subspaces $W_3^1 \times W_2^2$ defined by the systems of ODEs

$$\begin{aligned} L^1[y] &= y''' + a_2 y'' + a_1 y' + a_0 y = 0, \\ L^2[z] &= z'' + b_1 z' + b_0 z = 0. \end{aligned} \tag{43}$$

Using the invariant condition, we arrive at the following result.

Proposition 2.11. *If the nonlinear vector operator* $(F_1[u, v], F_2[u, v])$ *preserves the invariant subspace* $W_3^1 \times W_2^2$ *defined by system (43), then the coefficient functions in* $(F_1[u, v], F_2[u, v])$ *are given as follows:*

$$\begin{aligned}
f(u, v) &= (f_{14}v + f_{15})u + f_{22}v^3 + f_{23}v^2 + f_{24}v + f_{25}, \\
p(u, v) &= -2f_{14}u^2 + (-2f_{22}v^2 + p_{23}v + p_{24})u + p_{31}v^3 + p_{32}v^2 + p_{33}v + p_{34}, \\
r(u, v) &= (r_{14}v + r_{15})u^2 + (r_{21}v^4 + r_{22}v^3 + r_{23}v^2 + r_{24}v + r_{25})u \\
&\quad + r_{31}v^4 + r_{32}v^3 + r_{33}v^2 + r_{34}v + r_{35}, \\
g(u, v) &= (-2q_{11}v + g_{13})u + g_{21}v^2 + g_{22}v + g_{23}, \\
q(u, v) &= q_{11}v^2 + q_{12}v + q_{13}, \\
s(u, v) &= (s_{11}v^3 + s_{12}v^2 + s_{13}v + s_{14})u + s_{21}v^3 + s_{22}v^2 + s_{23}v + s_{24}.
\end{aligned}$$

The full description of the operators $(F_1[u, v], F_2[u, v])$ *admitting invariant subspaces defined by systems (43) is listed in table 5.*

2.10. $W_2^1 \times W_2^2$

Finally, we describe all vectors (F_1, F_2) admitting subspaces $W_2^1 \times W_2^2$ given by systems of the linear ODEs

$$\begin{aligned} L^1[y] &= y'' + a_1 y' + a_0 y = 0, \\ L^2[z] &= z'' + b_1 z' + b_0 z = 0. \end{aligned} \tag{44}$$

In this case, we have the following result.

Proposition 2.12. *If the nonlinear vector operator* $(F_1[u, v], F_2[u, v])$ *preserves the invariant subspace* $W_2^1 \times W_2^2$ *defined by system (44), then the coefficient functions in* $(F_1[u, v], F_2[u, v])$ *are given as follows:*

$$\begin{aligned}
f(u, v) &= (f_{13}v + f_{14})u^2 + (f_{22}v^2 + f_{23}v + f_{24})u - p_{13}v^3 + f_{32}v^2 + f_{33}v + f_{34}, \\
p(u, v) &= (p_{13}u + p_{14})v^2 + (-f_{22}u^2 + p_{23}u + p_{24})v - f_{13}u^3 + p_{32}u^2 + p_{33}u + p_{34}, \\
r(u, v) &= (r_{13}u + r_{14})v^3 + (r_{22}u^2 + r_{23}u + r_{24})v^2 \\
&\quad + (r_{31}u^3 + r_{32}u^2 + r_{33}u + r_{34})v + r_{41}u^3 + r_{42}u^2 + r_{43}u + r_{44}, \\
g(u, v) &= (g_{13}u + g_{14})v^2 + (g_{22}u^2 + g_{23}u + g_{24})v - q_{13}u^3 + g_{32}u^2 + g_{33}u + g_{34}, \\
q(u, v) &= (q_{13}v + q_{14})u^2 + (-g_{22}v^2 + q_{23}v + q_{24})u - g_{13}v^3 + q_{32}v^2 + q_{33}v + q_{34}, \\
s(u, v) &= (s_{13}u + s_{14})v^3 + (s_{22}u^2 + s_{23}u + s_{24})v^2 \\
&\quad + (s_{31}u^3 + s_{32}u^2 + s_{33}u + s_{34})v + s_{41}u^3 + s_{42}u^2 + s_{43}u + s_{44}.
\end{aligned}$$

In this case, the explicit description of vector operators $(F_1[u, v], F_2[u, v])$ *admitting invariant subspaces defined by systems (44) is given in table 6.*

Table 5. Operators $(F_1[u,v], F_2[u,v])$ admitting invariant subspace $W_3^1 \times W_2^2$ defined by system (43).

Parameters in (43)	Number	Operators $(F_1[u,v], F_2[u,v])$	System (43)
$b_1 = b_0 = 0$	1	$F_1 = [(f_{15}u + f_{25})u_x + (p_{24}u + p_{33}v + p_{34})v_x]_x + 2a_1 f_{15}u^2 + r_{25}u + r_{35}$, $F_2 = [(-q_{12}u + g_{21}v^2 + g_{22}v + g_{23})v_x + (q_{12}v + q_{13})u_x]_x + a_1(q_{12}v + q_{13})u + s_{23}v + s_{24}$	$L^1[y] = y''' + a_1 y' = 0$, $L^2[z] = z'' = 0$
	2	$F_1 = [f_{25}u_x + (p_{24}u + p_{32}v^2 + p_{33}v + p_{34})v_x]_x + r_{25}u + r_{34}v + r_{35}$, $F_2 = [(g_{21}v^2 + g_{22}v + g_{23})v_x + q_{13}u_x]_x - q_{13}a_2^2 u + s_{23}v + s_{24}$	$L^1[y] = y''' + a_2 y'' = 0$, $L^2[z] = z'' = 0$
	3	$F_1 = [(f_{14}uv + f_{15}u + f_{22}v^3 + f_{23}v^2 + f_{24}v + f_{25})u_x]_x + [(-2f_{14}u^2 - 2f_{22}v^2u + p_{23}vu$ $+ p_{24}u + p_{31}v^3 + p_{32}v^2 + p_{33}v + p_{34})v_x]_x + r_{25}u + r_{33}v^2 + r_{34}v + r_{35}$, $F_2 = [(-2q_{11}uv + g_{13}u + g_{21}v^2 + g_{22}v + g_{23})v_x]_x + [(q_{11}v^2 + q_{12}v + q_{13})u_x]_x + s_{23}v + s_{24}$	$L^1[y] = y''' = 0$, $L^2[z] = z'' = 0$
$b_1 = 0,\ b_0 \neq 0$	1	$F_1 = [(f_{15}u + f_{25})u_x + (p_{33}v + p_{34})v_x]_x + 2a_1 f_{15}u^2 + r_{25}u + 2b_0 p_{33}v^2 + b_0 p_{34}v + r_{35}$, $F_2 = [(-q_{12}u + g_{21}v^2 + g_{23})v_x + q_{12}vu_x]_x + q_{12}(a_1 - b_0)uv + 3b_0 g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + a_1 y' = 0$, $L^2[z] = z'' + b_0 z = 0$
	2	$F_1 = [(f_{15}u + f_{24}v + f_{25})u_x + (-\frac{3}{2}f_{24}u + p_{33}v + p_{34})v_x]_x$ $+ \frac{1}{2}b_0 f_{15}u^2 + (-\frac{3}{2}f_{24}b_0 v + r_{25})u + 2b_0 p_{33}v^2 + b_0 p_{34}v + r_{35}$, $F_2 = [(-q_{12}u + g_{21}v^2 + g_{23})v_x + q_{12}vu_x]_x - \frac{3}{4}b_0 q_{12}uv + 3b_0 g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + \frac{1}{4}b_0 y' = 0$, $L^2[z] = z'' + b_0 z = 0$
	3	$F_1 = [(f_{15}u + f_{25})u_x + (p_{32}v^2 + p_{33}v + p_{34})v_x]_x$ $+ 18b_0 f_{15}u^2 + r_{25}u + \frac{1}{3}b_0 p_{32}v^3 + 2b_0 p_{33}v^2 + b_0 p_{34}v + r_{35}$, $F_2 = [(-q_{12}u + g_{21}v^2 + g_{23})v_x + q_{12}vu_x]_x + 8b_0 q_{12}uv + 3b_0 g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + 9b_0 y' = 0$, $L^2[z] = z'' + b_0 z = 0$
	4	$F_1 = [(f_{15}u + f_{23}v^2 + f_{25})u_x + (p_{23}uv + p_{31}v^3 + p_{33}v + p_{34})v_x]_x$ $+ 8b_0 f_{15}u^2 + [4b_0(2f_{23} + p_{23})v^2 + r_{25}]u + 4p_{31}b_0 v^4 + r_{33}v^2 + p_{34}b_0 v + r_{35}$, $F_2 = [(g_{13}u + g_{21}v^2 + g_{23})v_x + q_{12}vu_x]_x + 3b_0(2q_{12} + g_{13})uv + 3b_0 g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + 4b_0 y' = 0$, $L^2[z] = z'' + b_0 z = 0$
	5	$F_1 = [(f_{15}u + f_{25})u_x + (p_{31}v^3 + p_{33}v + p_{34})v_x]_x$ $+ 32b_0 f_{15}u^2 + r_{25}u + b_0 p_{31}v^4 + 2b_0 p_{33}v^2 + b_0 p_{34}v + r_{35}$, $F_2 = [(-q_{12}u + g_{21}v^2 + g_{23})v_x + q_{12}vu_x]_x + 15b_0 q_{12}uv + 3b_0 g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + 16b_0 y' = 0$, $L^2[z] = z'' + b_0 z = 0$
	6	$F_1 = [(f_{14}uv + f_{15}u + f_{23}v^2 + f_{24}v + f_{25})u_x]_x + [(-2f_{14}u^2 - 3f_{23}uv + p_{24}u + p_{32}v^2 + p_{33}v$ $+ p_{34})v_x]_x + b_0(-3f_{14}v + 2f_{15})u^2 + [-6f_{23}b_0 v^2 + 2b_0(p_{24} + f_{24})v + r_{25}]u + 3p_{32}b_0 v^3$ $+ 2b_0 p_{33}v^2 + r_{34}v + r_{35}$, $F_2 = [(-q_{12}u + g_{21}v^2 + g_{23})v_x + (q_{12}v + q_{13})u_x]_x + 3b_0 g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + b_0 y' = 0$, $L^2[z] = z'' + b_0 z = 0$
	7	$F_1 = [f_{25}u_x + (p_{32}v^2 + p_{34})v_x]_x + r_{25}u + 3b_0^2 p_{32}v^3 + r_{34}v$, $F_2 = [(g_{21}v^2 + g_{23})v_x + q_{13}u_x]_x - a_2^2 q_{13}u + 3b_0 g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + a_2 y'' + b_0 y' + a_2 b_0 y = 0$, $L^2[z] = z'' + b_0 z = 0$
	8	$F_1 = [(f_{23}v^2 + f_{25})u_x + (-6f_{23}uv + p_{32}v^2 + p_{34})v_x]_x + (\frac{5}{3}f_{23}a_2^2 v^2 + r_{25})u - \frac{1}{3}p_{32}a_2^2 v^3 + r_{34}v$, $F_2 = [(g_{21}v^2 + g_{23})v_x + q_{13}u_x]_x - q_{13}a_2^2 u - \frac{1}{3}a_2^2 g_{21}v^3 + g_{23}v$	$L^1[y] = y''' + a_2 y'' - \frac{1}{9}a_2^2 y' - \frac{1}{9}a_2^3 y = 0$, $L^2[z] = z'' - \frac{1}{9}a_2^2 z = 0$

Table 5. (Continued.)

Parameters in (43)	Number	Operators ($F_1[u, v]$, $F_2[u, v]$)	System (43)
$b_1 = 0,\ b_0 \neq 0$	9	$F_1 = [f_{25}u_x + (p_{33}v + p_{34})v_x]_x + r_{25}u - \frac{1}{2}p_{33}a_2^2v^2 - \frac{1}{4}a_2^2p_{34}v + r_{35}$, $F_2 = [(-3q_{12}u + g_{21}v^2 + g_{23})v_x + q_{12}vu_x]_x + \frac{3}{4}q_{12}a_2^2uv - \frac{3}{4}a_2^2g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + a_2y'' = 0$, $L^2[z] = z'' - \frac{1}{4}a_2^2z = 0$
	10	$F_1 = [f_{25}u_x + (p_{32}v^2 + p_{34})v_x]_x + r_{25}u - \frac{1}{3}p_{32}a_2^2v^3 - a_2^2p_{34}v$, $F_2 = [(g_{21}v^2 + g_{23})v_x + q_{13}u_x]_x - 9q_{13}a_2^2u - 3a_2^2g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + a_2y'' - 9a_2^2y' - 9a_2^3y = 0$, $L^2[z] = z'' - a_2^2z = 0$
	11	$F_1 = [f_{25}u_x + p_{34}v_x]_x + r_{25}u - a_2^2p_{34}v$, $F_2 = [(g_{21}v^2 + g_{23})v_x + q_{13}u_x]_x + q_{13}a_1u - 3a_2^2g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + a_2y'' + a_1y' + a_1a_2y = 0$, $L^2[z] = z'' - a_2^2z = 0$
	12	$F_1 = [f_{25}u_x + (p_{33}v + p_{34})v_x]_x + r_{25}u - a_2^2p_{34}v$, $F_2 = [(g_{21}v^2 + g_{23})v_x + q_{13}u_x]_x - 4q_{13}a_2^2u - 3a_2^2g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + a_2y'' - 4a_2^2y' - 4a_2^3y = 0$, $L^2[z] = z'' - a_2^2z = 0$
	13	$F_1 = [f_{25}u_x + (p_{33}v + p_{34})v_x]_x + r_{25}u - 2p_{33}a_2^2v^2 - a_2^2p_{34}v + r_{35}$, $F_2 = [(g_{21}v^2 + g_{23})v_x + q_{13}u_x]_x - 3a_2^2g_{21}v^3 + s_{23}v$	$L^1[y] = y''' + a_2y'' = 0$, $L^2[z] = z'' - a_2^2z = 0$
$b_1 \neq 0,\ b_0 = 0$	1	$F_1 = [(f_{15}u + f_{25})u_x + p_{34}v_x]_x + 2a_1f_{15}u^2 + r_{25}u - p_{34}b_1^2v + r_{35}$, $F_2 = [(g_{22}v + g_{23})v_x + q_{13}u_x]_x + q_{13}a_1u - 2g_{22}b_1^2v^2 + s_{23}v + s_{24}$	$L^1[y] = y''' + a_1y' = 0$, $L^2[z] = z'' + b_1z' = 0$
	2	$F_1 = [(f_{15}u + f_{24}v + f_{25})u_x - (\frac{3}{2}f_{24}u + p_{34})v_x]_x$ $\quad - \frac{1}{2}b_1^2f_{15}u^2 + (\frac{3}{2}f_{24}b_1^2v + r_{25})u + b_1^2p_{34}v + r_{35}$, $F_2 = [(g_{22}v + g_{23})v_x + q_{13}u_x]_x - \frac{1}{4}q_{13}b_1^2u - 2g_{22}b_1^2v^2 + s_{23}v + s_{24}$	$L^1[y] = y''' - \frac{1}{4}b_1^2y' = 0$, $L^2[z] = z'' + b_1z' = 0$
	3	$F_1 = [(f_{15}u + f_{25})u_x + (p_{33}v + p_{34})v_x]_x - 8b_1^2f_{15}u^2 + r_{25}u - \frac{1}{2}b_1^2p_{33}v^2 - b_1^2p_{34}v + r_{35}$, $F_2 = [(g_{22}v + g_{23})v_x + q_{13}u_x]_x - 4q_{13}b_1^2u - 2g_{22}b_1^2v^2 + s_{23}v + s_{24}$	$L^1[y] = y''' - 4b_1^2y' = 0$, $L^2[z] = z'' + b_1z' = 0$
	4	$F_1 = [(f_{15}u + f_{24}v + f_{25})u_x + (p_{22}u + p_{33}v + p_{34})v_x]_x - 2b_1^2f_{15}u^2$ $\quad + [-2b_1^2(f_{24} + p_{24})v + r_{25}]u - 2b_1^2p_{33}v^2 + r_{34}v + r_{35}$, $F_2 = [(g_{13}u + g_{22}v + g_{23})v_x + (-2g_{13}v + q_{13})u_x]_x + b_1^2(2g_{13}v - q_{13}) - 2g_{22}b_1^2v^2 + s_{23}v + s_{24}$	$L^1[y] = y''' - b_1^2y' = 0$, $L^2[z] = z'' + b_1z' = 0$
	5	$F_1 = [(b_1p_{24} + f_{25})u_x + (-p_{24}(a_2 + b_1)u - p_{33}v + p_{34})v_x]_x$ $\quad + (2a_1b_1^2p_{24}v + r_{25})u + 2b_1^2p_{33}v^2 + r_{34}v + r_{35}$, $F_2 = [(g_{22}v + g_{23})v_x + q_{13}u_x]_x - q_{13}(a_2 - b_1)^2u - 2b_1^2g_{22}v^2 + s_{23}v + s_{24}$	$L^1[y] = y''' + a_2y'' + b_1(a_2 - b_1)y' = 0$, $L^2[z] = z'' + b_1z' = 0$
	6	$F_1 = [f_{25}u_x + p_{34}v_x]_x + r_{25}u$, $F_2 = [(g_{22}v + g_{23})v_x + q_{13}u_x]_x + q_{13}a_1u - 2b_1^2g_{22}v^2 + s_{23}v + s_{24}$	$L^1[y] = y''' + b_1y'' + a_1y' + a_1b_1y = 0$, $L^2[z] = z'' + b_1z' = 0$
	7	$F_1 = [(f_{24}v + f_{25})u_x + (-f_{24}u + p_{33}v + p_{34})v_x]_x + (-3b_1^2f_{24}v + r_{25})u$, $F_2 = [(g_{22}v + g_{23})v_x + q_{13}u_x]_x - 4q_{13}b_1^2u - 2b_1^2g_{22}v^2 + s_{23}v + s_{24}$	$L^1[y] = y''' + b_1y'' - 4b_1^2y' - 4b_1^3y = 0$, $L^2[z] = z'' + b_1z' = 0$
	8	$F_1 = [(f_{24}v + f_{25})u_x + (-\frac{5}{2}f_{24}u + p_{34})v_x]_x + (3b_1^2f_{24}v + r_{25})u$, $F_2 = [(g_{22}v + g_{23})v_x + q_{13}u_x]_x - \frac{1}{4}q_{13}b_1^2u - 2b_1^2g_{22}v^2 + s_{23}v + s_{24}$	$L^1[y] = y''' + b_1y'' - \frac{1}{4}b_1^2y' - \frac{1}{4}b_1^3y = 0$, $L^2[z] = z'' + b_1z' = 0$

Table 5. (Continued.)

Parameters in (43)	Number	Operators $(F_1[u, v], F_2[u, v])$	System (43)
$b_1 \neq 0,\ b_0 = 0$	9	$F_1 = [(f_{24}v + f_{25})u_x + (p_{24}u + p_{32}v^2 + p_{33}v + p_{34})v_x]_x$ $+ [-3b_1^2(2f_{24} + p_{24})v + r_{25}]u - 3b_1^2 p_{32}v^3 + r_{33}v^2 + r_{34}v + r_{35}$, $F_2 = [(g_{22}v + g_{23})v_x + q_{13}u_x]_x - 4q_{13}b_1^2 u - 2b_1^2 g_{22}v^2 + s_{23}v + s_{24}$	$L^1[y] = y''' + 3b_1y'' + 2b_1^2 y' = 0$, $L^2[z] = z'' + b_1z' = 0$
	10	$F_1 = [(-\frac{1}{5}p_{24}v + f_{25})u_x + (p_{24}u + p_{32}v^2 + p_{33}v + p_{34})v_x]_x$ $+ (-\frac{8}{5}b_1^2 p_{24}v + r_{25})u - \frac{4}{3}b_1^2 p_{32}v^3 - 2b_1^2 p_{33}v^2 + r_{34}v + r_{35}$, $F_2 = [(g_{22}v + g_{23})v_x + q_{13}u_x]_x - 9q_{13}b_1^2 u - 2b_1^2 g_{22}v^2 + s_{23}v + s_{24}$	$L^1[y] = y''' + 4b_1y'' + 3b_1^2 y' = 0$, $L^2[z] = z'' + b_1z' = 0$
$b_1 \neq 0,\ b_0 \neq 0$	1	$F_1 = [(f_{15}u + f_{25})u_x + p_{34}v_x]_x - \frac{2}{9}b_1^2 f_{15}u^2 + r_{25}u - \frac{4}{9}b_1^2 p_{34}v^3 + r_{35}$, $F_2 = [(g_{13}u + g_{23})v_x - 2g_{13}vu_x]_x + s_{23}v$	$L^1[y] = y''' - \frac{1}{9}b_1^2 y' = 0$, $L^2[z] = z'' + b_1z' + \frac{2}{9}b_1^2 z = 0$
	2	$F_1 = [f_{25}u_x + (p_{33}v + p_{34})v_x]_x + r_{25}u - \frac{8}{9}b_1^2 p_{33}v^2 + r_{34}v$, $F_2 = [g_{23}v_x + q_{13}u_x]_x - q_{13}b_1^2 u + s_{23}v$	$L^1[y] = y''' + 2b_1y'' + \frac{11}{9}b_1^2 y' + \frac{2}{9}b_1^3 y = 0$, $L^2[z] = z'' + b_1z' + \frac{2}{9}b_1^2 z = 0$
	3	$F_1 = [f_{25}u_x + (9p_{33}v + p_{34})v_x]_x + r_{25}u - 8b_1^2 p_{33}v^2 - \frac{1}{9}b_1^2 p_{34}v$, $F_2 = [g_{23}v_x + q_{13}u_x]_x - q_{13}b_1^2 u + s_{23}v$	$L^1[y] = y''' + \frac{2}{3}b_1y'' - b_1^2 y' - \frac{2}{3}b_1^3 y = 0$, $L^2[z] = z'' + b_1z' + \frac{2}{9}b_1^2 z = 0$
	4	$F_1 = [f_{25}u_x + (2p_{33}v + p_{34})v_x]_x + r_{25}u - b_1^2 p_{33}v^2 - \frac{1}{9}b_1^2 p_{34}v$, $F_2 = [g_{23}v_x + q_{13}u_x]_x - \frac{16}{9}q_{13}b_1^2 u + s_{23}v$	$L^1[y] = y''' + \frac{2}{3}b_1y'' - \frac{16}{9}b_1^2 y' - \frac{32}{27}b_1^3 y = 0$, $L^2[z] = z'' + b_1z' + \frac{2}{9}b_1^2 z = 0$
	5	$F_1 = [f_{25}u_x + (p_{33}v + p_{34})v_x]_x + r_{25}u - \frac{1}{2}b_1^2 p_{33}v^2 + r_{34}v$, $F_2 = [g_{23}v_x + q_{13}u_x]_x - \frac{16}{9}q_{13}b_1^2 u + s_{23}v$	$L^1[y] = y''' + \frac{7}{3}b_1y'' + \frac{14}{9}b_1^2 y' + \frac{8}{27}b_1^3 y = 0$, $L^2[z] = z'' + b_1z' + \frac{2}{9}b_1^2 z = 0$
	6	$F_1 = [(f_{15}u + f_{25})u_x + p_{32}v^2 v_x]_x - 18b_1^2 f_{15}u^2 + r_{25}u - 12b_1^2 p_{32}v^3 + r_{35}$, $F_2 = [(g_{13}u + g_{23})v_x - 2g_{13}vu_x]_x + 20b_1^2 g_{13}uv + s_{23}v$	$L^1[y] = y''' - 9b_1^2 y' = 0$, $L^2[z] = z'' + b_1z' - 2b_1^2 z = 0$
	7	$F_1 = [f_{25}u_x + p_{34}v_x]_x + r_{25}u - 9b_1^2 p_{34}v + r_{35}$, $F_2 = [(g_{13}u + g_{23})v_x - \frac{1}{2}g_{13}vu_x]_x - 4b_1^2 g_{13}uv + s_{23}v$	$L^1[y] = y''' + 3b_2y'' - 10b_1^2 y' = 0$, $L^2[z] = z'' + b_1z' - 6b_1^2 z = 0$
	8	$F_1 = [f_{25}u_x + (2p_{33}v + p_{34})v_x]_x + r_{25}u - p_{33}b_1^2 v^2 - \frac{9}{4}b_1^2 p_{34}v$, $F_2 = [g_{23}v_x + q_{13}u_x]_x - 9b_1^2 q_{13}u + s_{23}v$	$L^1[y] = y''' - \frac{1}{2}b_1y'' - 9b_1^2 y' + \frac{9}{2}b_1^3 y = 0$, $L^2[z] = z'' + b_1z' - \frac{3}{4}b_1^2 z = 0$
	9	$F_1 = [f_{25}u_x + (2p_{33}v + p_{34})v_x]_x + r_{25}u - 9p_{33}b_1^2 v^2 - \frac{9}{4}b_1^2 p_{34}v$, $F_2 = [g_{23}v_x + q_{13}u_x]_x - b_1^2 q_{13}u + s_{23}v$	$L^1[y] = y''' - \frac{1}{2}b_1y'' - b_1^2 y' + \frac{1}{2}b_1^3 y = 0$, $L^2[z] = z'' + b_1z' - \frac{3}{4}b_1^2 z = 0$
	10	$F_1 = [f_{25}u_x + (2p_{33}v + p_{34})v_x]_x + r_{25}u - p_{33}b_1^2 v^2 - \frac{1}{4}b_1^2 p_{34}v$, $F_2 = [g_{23}v_x + q_{13}u_x]_x - 9b_1^2 q_{13}u + s_{23}v$	$L^1[y] = y''' + \frac{3}{2}b_1y'' - 9b_1^2 y' - \frac{27}{2}b_1^3 y = 0$, $L^2[z] = z'' + b_1z' - \frac{3}{4}b_1^2 z = 0$
	11	$F_1 = [f_{25}u_x + (p_{33}v + p_{34})v_x]_x + r_{25}u - \frac{1}{2}p_{33}b_1^2 v^2 + r_{34}v$, $F_2 = [g_{23}v_x + q_{13}u_x]_x - 9b_1^2 q_{13}u + s_{23}v$	$L^1[y] = y''' + 4b_1y'' + \frac{9}{4}b_1^2 y' - \frac{9}{4}b_1^3 y = 0$, $L^2[z] = z'' + b_1z' - \frac{3}{4}b_1^2 z = 0$

Table 6. Operators $(F_1[u,v], F_2[u,v])$ admitting invariant subspace $W_2^1 \times W_2^2$ defined by system (44).

Parameters in (44)	Number	Operators $(F_1[u,v], F_2[u,v])$	System (44)
$a_1 = a_0 = 0$	1	$F_1 = [(f_{14}u^2 + f_{24}u - \frac{1}{2}p_{23}v^2 - p_{33}v + f_{34})u_x + (p_{23}uv + p_{24}v + p_{33}u + p_{34})v_x]_x$ $+ 2b_0(p_{23}u + p_{24})v^2 + b_0(p_{33}u + p_{34})v + r_{43}u + r_{44}$ $F_2 = [(g_{14}v^2 + g_{32}u^2 + g_{33}u + g_{34})v_x + (-2g_{32}uv + q_{33}v + q_{34})u_x]_x$ $+ 3b_0g_{14}v^3 + (b_0g_{32}u^2 + b_0g_{33}u + s_{34})v$	$L^1[y] = y'' = 0,$ $L^2[z] = z'' + b_0z = 0$
	2	$F_1 = \{[(f_{13}v + f_{14})u^2 + (f_{22}v^2 + f_{23}v + f_{24})u - p_{13}v^3 + f_{32}v^2 + f_{33}v + f_{34}]u_x$ $+ [(p_{13}u + p_{14})v^2 + (-f_{22}u^2 + p_{23}u + p_{24})v - f_{13}u^3 + p_{32}u^2 + p_{33}u + p_{34}]v_x\}_x$ $+ r_{34}v + r_{43}u + r_{44},$ $F_2 = \{[(g_{13}u + g_{14})v^2 + (g_{22}u^2 + g_{23}u + g_{24})v - q_{13}u^3 + g_{32}u^2 + g_{33}u + g_{34}]v_x$ $+ [(q_{13}v + q_{14})u^2 + (-g_{22}v^2 + q_{23}v + q_{24})u - g_{13}v^3 + q_{32}v^2 + q_{33}v + q_{34}]u_x\}_x$ $+ s_{34}v + s_{43}u + s_{44}$	$L^1[y] = y'' = 0,$ $L^2[z] = z'' = 0$
$a_1 = 0,\ a_0 \neq 0$	1	$F_1 = [(f_{14}u^2 - \frac{1}{2}p_{23}v^2 - p_{33}v + f_{34})u_x + (p_{23}uv + p_{33}u + p_{34})v_x]_x$ $+ p_{23}(2b_0 - \frac{1}{2}a_0)uv^2 + [p_{33}(b_0 - a_0)u + p_{34}b_0]v + 3a_0f_{14}u^3 + r_{43}u,$ $F_2 = [(g_{14}v^2 - \frac{1}{2}q_{23} - q_{33}u + g_{34})v_x + (q_{23}uv + q_{33}v + q_{34})u_x]_x$ $+ 3b_0g_{14}v^3 + [q_{23}(2a_0 - \frac{1}{2}b_0)u^2 + q_{33}(a_0 - b_0)u + q_{34}a_0]v + s_{43}u$	$L^1[y] = y'' + a_0y = 0,$ $L^2[z] = z'' + b_0z = 0$
	2	$F_1 = [(f_{14}u^2 + f_{23}uv - \frac{1}{2}p_{23}v^2 - p_{33}v + f_{34})u_x]_x + [(p_{23}uv - \frac{5}{6}f_{23}u^2 + p_{33}u + p_{34})v_x]_x$ $+ \frac{35}{2}p_{23}a_0uv^2 + 9a_0(-\frac{5}{6}f_{23}u^2 + \frac{8}{9}p_{33}u + p_{34})v + 3a_0f_{14}u^3 + r_{43}u,$ $F_2 = [(g_{14}v^2 - \frac{1}{2}q_{23}u^2 - q_{33}u + g_{34})v_x + (q_{14}u^2 + q_{23}uv + q_{33}v + 9q_{34})u_x]_x$ $+ 27a_0g_{14}v^3 + (-\frac{5}{2}a_0q_{23}u^2 - 8a_0q_{33}u + s_{34})v + \frac{1}{3}a_0q_{14}u^3 + 9a_0q_{34}u$	$L^1[y] = y'' + a_0y = 0,$ $L^2[z] = z'' + 9a_0z = 0$
	3	$F_1 = [(f_{13}vu^2 + f_{14}u^2 + f_{32}v^2 + f_{33}v + f_{34})u_x]_x + [(-2f_{32}uv - f_{13}u^3 + p_{33}u + p_{34})v_x]_x$ $- 15a_0f_{32}uv^2 + a_0[-5f_{13}u^3 + 3(f_{33} + 2p_{33})u + 4p_{34}]v + 3a_0f_{14}u^3 + r_{43}u,$ $F_2 = [(g_{14}v^2 + g_{32}u^2 - q_{33}u + g_{34})v_x + (-2g_{32}vu + q_{24}u + q_{33}v + q_{34})u_x]_x$ $+ 12a_0g_{14}v^3 + (-3q_{33}a_0u + s_{34})v + a_0q_{34}u$	$L^1[y] = y'' + a_0y = 0,$ $L^2[z] = z'' + 4a_0z = 0$
	4	$F_1 = [(f_{14}u^2 + f_{23}uv + f_{32}v^2 - p_{33}v + f_{34})u_x]_x + [(p_{14}v^2 + p_{23}uv + p_{32}u^2 + p_{33}u + p_{34})v_x]_x$ $+ 3p_{14}a_0v^3 + 3a_0(p_{23} + f_{32})uv^2 + [3a_0(p_{32} + f_{23})u^2 + r_{34}]v + 3a_0f_{14}u^3 + r_{43}u,$ $F_2 = [(g_{14}v^2 + g_{23}uv + g_{32}u^2 - g_{33}u + g_{34})v_x]_x + [(q_{14}u^2 + q_{23}uv + q_{32}v^2 + q_{33}v + q_{34})u_x]_x$ $+ 3q_{14}a_0v^3 + 3a_0(q_{32} + g_{23})uv^2 + [3a_0(g_{32} + q_{23})u^2 + s_{34}]v + 3a_0q_{14}u^3 + s_{43}u$	$L^1[y] = y'' + a_0y = 0,$ $L^2[z] = z'' + a_0z = 0$

Table 6. (Continued.)

Parameters in (44)	Number	Operators ($F_1[u,v]$, $F_2[u,v]$)	System (44)
$a_1 \neq 0,\ a_0 = 0$	1	$F_1 = [(f_{24}u + f_{34})u_x + (p_{24}v + p_{34})v_x]_x + 2p_{24}b_0v^2 + b_0p_{34}v - 2f_{24}a_1^2u^2 + r_{43}u + r_{44}$, $F_2 = [(g_{14}v^2 - q_{33}u + g_{34})v_x + q_{33}vu_x]_x + 3b_0g_{14}v^3 + [-q_{33}(a_1^2 + b_0)u + s_{34}]v$	$L^1[y] = y'' + a_1y' = 0$, $L^2[z] = z'' + b_0z = 0$
	2	$F_1 = [(f_{24}u + f_{32}v^2 + f_{34})u_x + (-4f_{32}uv + p_{24}v + p_{34})v_x]_x$ $+ a_1^2(2f_{32}u - \frac{1}{2}p_{24})v^2 - \frac{1}{4}p_{34}a_1^2v - 2f_{24}a_1^2v + r_{43}u + r_{44}$, $F_2 = [(g_{14}v^2 + g_{33}v + g_{34})v_x + q_{33}vu_x]_x - \frac{3}{4}g_{14}a_1^2v^3 + [-\frac{3}{2}(\frac{1}{2}g_{33} + q_{22})u + s_{34}]v$	$L^1[y] = y'' + a_1y' = 0$, $L^2[z] = z'' - \frac{1}{4}a_1^2z = 0$
	3	$F_1 = [(f_{24}u + f_{33}v + f_{34})u_x + (p_{24}v - 2f_{33}u + p_{34})v_x]_x$ $- 2p_{24}a_1^2v^2 + a_1^2(2f_{33}u - p_{34})v - 2f_{24}a_1^2u^2 + r_{43}u + r_{44}$, $F_2 = [(g_{14}v^2 - q_{33}u + g_{24})v_x + (q_{33}v + q_{34})u_x]_x - 3a_1^2g_{14}v^3 + s_{14}v$	$L^1[y] = y'' + a_1y' = 0$, $L^2[z] = z'' - a_1^2z = 0$
	4	$F_1 = [(f_{24}u + f_{34})u_x + (p_{24}v + p_{33}u + p_{34})v_x]_x - 2f_{24}a_1^2u^2 + r_{43}u + r_{44}$, $F_2 = [(g_{14}v^2 + g_{24}v - q_{33}u + g_{34})v_x + (q_{33}v + q_{34})u_x]_x + (-q_{33}a_1^2u + s_{34})v - q_{34}a_1^2u + s_{44}$	$L^1[y] = y'' + a_1y' = 0$, $L^2[z] = z'' = 0$
	5	$F_1 = [(\frac{1}{2}p_{32}uv + f_{24}u + f_{33}v + f_{34})u_x + (p_{32}u^2 + p_{33}u + p_{34})v_x]_x$ $- a_1^2(p_{32}u^2 + p_{33}u + p_{34})v - 2a_1^2f_{24}u^2 + r_{43}u + r_{44}$, $F_2 = [(\frac{1}{2}q_{32}uv + g_{24}v + g_{33}u + g_{34})v_x + (q_{32}v^2 + q_{33}v + q_{34})u_x]_x$ $- a_1^2(q_{32}u + 2g_{24})v^2 + (-q_{33}a_1^2u + s_{34})v - q_{34}a_1^2u + s_{44}$	$L^1[y] = y'' + a_1y' = 0$, $L^2[z] = z'' - a_1z' = 0$
	6	$F_1 = [(f_{24}u + p_{33}b_1v + f_{34})u_x + (-p_{33}a_1u - p_{33}b_1u + p_{34})v_x]_x$ $+ [p_{33}b_1^2(a_1 + b_1)u - p_{34}b_1^2]v - 2a_1^2f_{24}u^2 + r_{43}u + r_{44}$, $F_2 = [(g_{24}v + q_{33}a_1u + g_{34})v_x + (-q_{33}a_1v - q_{33}b_1v + q_{34})u_x]_x$ $- 2b_1^2g_{24}v^2 + [a_1^2g_{33}(a_1 + b_1)u + s_{34}]v - q_{34}a_1^2u + s_{44}$	$L^1[y] = y'' + a_1y' = 0$, $L^2[z] = z'' + b_1z' = 0$
	7	$F_1 = [(f_{24}u + f_{33}v + f_{34})u_x + (p_{24}v - 3f_{33}u - 4p_{34})v_x]_x$ $- \frac{1}{8}p_{24}a_1^2v^2 + a_1^2(\frac{3}{4}f_{33}u + p_{34})v - 2a_1^2f_{24}u^2 + r_{43}u + r_{44}$, $F_2 = [(g_{24}v - \frac{2}{3}q_{33}u + g_{34})v_x + (q_{33}v + q_{34})u_x]_x$ $- \frac{1}{2}a_1^2g_{24}v^2 + (-q_{33}a_1^2u + s_{34})v - q_{34}a_1^2u + s_{44}$	$L^1[y] = y'' + a_1y' = 0$, $L^2[z] = z'' + \frac{1}{2}a_1z' = 0$
	8	$F_1 = [(f_{24}u + f_{33}v + f_{34})u_x + (p_{24}v + p_{33}u + p_{34})v_x]_x$ $- 2p_{24}a_1^2u^2 + [-2a_1^2(f_{33} + p_{33})u + r_{34}]v - 2a_1^2f_{24}u^2 + r_{43}u + r_{44}$, $F_2 = [(g_{24}u + g_{33}v + g_{34})u_x + (q_{24}v + q_{33}u + q_{34})v_x]_x$ $- 2g_{24}a_1^2v^2 + [-2a_1^2(g_{33} + q_{33})u + s_{34}]v - 2a_1^2q_{24}u^2 + s_{43}u + s_{44}$	$L^1[y] = y'' + a_1y' = 0$, $L^2[z] = z'' + a_1z' = 0$

Table 6. (Continued.)

Parameters in (44)	Number	Operators ($F_1[u,v]$, $F_2[u,v]$)	System (44)
$a_1 \neq 0,\ a_0 \neq 0$	1	$F_1 = [(f_{32}v^2 + f_{34})u_x + (-4f_{32}uv + p_{34})v_x]_x + \frac{5}{16}a_1^2 f_{32}uv^2 - \frac{1}{16}p_{34}a_1^2 v + r_{43}u,$ $F_2 = [(g_{14}v^2 + g_{34})v_x + q_{34}u_x]_x - \frac{3}{16}a_1^2 g_{14}v^3 + s_{34}v - \frac{9}{16}q_{34}a_1^2 u$	$L^1[y] = y'' + a_1 y' + \frac{3}{16}a_1^2 y = 0,$ $L^2[z] = z'' - \frac{1}{16}a_1^2 z = 0$
	2	$F_1 = [(f_{33}v + f_{34})u_x + (-2f_{33}u + p_{34})v_x]_x - \frac{1}{9}p_{34}a_1^2 v + r_{43}u,$ $F_2 = [(g_{14}v^2 + g_{34})v_x + q_{34}u_x]_x - \frac{1}{3}a_1^2 g_{14}v^3 + s_{34}v - \frac{4}{9}q_{34}a_1^2 u$	$L^1[y] = y'' + a_1 y' + \frac{2}{9}a_1^2 y = 0,$ $L^2[z] = z'' - \frac{1}{9}a_1^2 z = 0$
	3	$F_1 = [f_{34}u_x + p_{34}v_x]_x - 9p_{34}a_1^2 v + r_{43}u,$ $F_2 = [(g_{14}v^2 + g_{34})v_x + q_{24}uu_x]_x - 27a_1^2 g_{14}v^3 + s_{34}v - \frac{1}{2}q_{24}a_1^2 u^2$	$L^1[y] = y'' + a_1 y' - \frac{3}{4}a_1^2 y = 0,$ $L^2[z] = z'' - 9a_1^2 z = 0,$
	4	$F_1 = [(f_{32}v^2 + f_{34})u_x + (-4f_{32}uv + p_{34})v_x]_x + \frac{35}{4}a_1^2 f_{32}uv^2 - p_{34}a_1^2 v + r_{43}u,$ $F_2 = [(g_{14}v^2 + g_{34})v_x + q_{24}uu_x]_x - 3a_1^2 g_{14}v^3 + s_{34}v - \frac{9}{2}q_{24}a_1^2 u^2$	$L^1[y] = y'' + a_1 y' - \frac{3}{4}a_1^2 y = 0$ $L^2[z] = z'' - a_1^2 z = 0,$
	5	$F_1 = [(f_{33}v + f_{34})u_x - \frac{4}{3}f_{33}uv_x]_x + \frac{45}{4}a_1^2 f_{33}uv + r_{43}u,$ $F_2 = [(g_{24}v + g_{34})v_x + q_{24}uu_x]_x - 18a_1^2 s_{24}v^2 + s_{34}v - \frac{1}{2}q_{24}a_1^2 u^2 + s_{44}$	$L^1[y] = y'' + a_1 y' - \frac{3}{4}a_1^2 y = 0,$ $L^2[z] = z'' + 3a_1 z' = 0$
	6	$F_1 = [(f_{33}v + f_{34})u_x + (-p_{33}a_1 u - p_{33}b_1 u + p_{34})v_x]_x + 2p_{33}a_1 b_1^2 uv + r_{43}u,$ $F_2 = [(g_{24}v + g_{34})v_x + q_{34}u_x]_x - 2b_1^2 g_{24}v^2 + s_{34}v - q_{33}(a_1 - b_1)^2 u + s_{44}$	$L^1[y] = y'' + a_1 y' + b_1(a_1 - b_1)y = 0,$ $L^2[z] = z'' + b_1 z' = 0$
	7	$F_1 = [(f_{33}v + f_{34})u_x + (p_{24}v + p_{33}u + p_{34})v_x]_x - \frac{1}{3}a_1^2(2f_{33} + p_{33})uv + r_{43}u,$ $F_2 = [(g_{24}v + g_{34})v_x + q_{34}u_x]_x - \frac{2}{9}a_1^2 g_{24}v^2 + s_{34}v - \frac{4}{9}a_1^2 q_{34}u + s_{44}$	$L^1[y] = y'' + a_1 y' + \frac{2}{9}a_1^2 y = 0,$ $L^2[z] = z'' + \frac{1}{3}a_1 z' = 0$
	8	$F_1 = [(f_{33}v + f_{34})u_x - \frac{7}{6}f_{33}uv_x]_x + 40a_1^2 f_{33}uv + r_{43}u,$ $F_2 = [(g_{24}v + g_{34})v_x + q_{14}u^2 v_x]_x - 72a_1^2 g_{24}v^2 + s_{34}v - 3q_{14}a_1^2 u^3 + s_{44}$	$L^1[y] = y'' + a_1 y' - 2a_1^2 y = 0,$ $L^2[z] = z'' + 6a_1 z' = 0$

3. Examples of exact solutions of systems (1)

In this section, we provide several examples to illustrate how to construct exact solutions of nonlinear system (1) by using the results in section 2. The algorithm for deriving exact solutions and reductions to finite-dimensional dynamical systems for equations (1) compatible with (9) are given as follows.

(i) Determine the coefficient functions in (1) and a_j and b_k in $L^1[y] = 0$ and $L^2[z] = 0$ from the invariant condition (10).

(ii) Solving (9) leads to the invariant subspaces.

(iii) Construct exact solutions of (1) on invariant subspaces.

(iv) Substituting the exact solution into equation (1) yields reduction to a finite-dimensional system for expansion coefficient functions.

Example 1. From the fifth result in table 4, the system

$$\begin{aligned} u_t &= [uu_x + v_x]_x - v, \\ v_t &= v_{xx} + u_{xx} \end{aligned} \tag{45}$$

has the exact solution of the form

$$u = c_1(t) + c_2(t)x + c_3(t)x^2, \qquad v = d_1(t) + d_2(t)x + d_3(t)\exp(-x),$$

where $c_i(t)$ and $d_i(t)$ satisfy the system of ODEs:

$$\begin{aligned} c_1' &= c_2^2 + 2c_1c_3 - d_1, \qquad & c_2' &= 6c_2c_3 - d_2, \\ c_3' &= 6c_3^2, & d_1' &= 2c_3, \\ d_2' &= 0, & d_3' &= d_3. \end{aligned}$$

Solving this dynamic system, we obtain the solution of system (45)

$$\begin{aligned} u = {} & \frac{1}{320(6t+c_3)}\big[144d_2^2t^4 + 96c_3d_2^2t^3 + \big(-240c_2d_2 + 4c_3^2d_2^2\big)t^2 \\ & + \big(-80c_2c_3d_2 - 4c_3^3d_2^2\big)t - 80c_2^2 - 20c_2c_3^2d_2 - c_3^4d_2^2\big] - \frac{1}{8}d_1(6t+c_3) \\ & + \frac{1}{24}(6t+c_3)\ln|6t+c_3| - \frac{6t+c_3}{32} + c_1(6t+c_3)^{-\frac{1}{3}} \\ & - \frac{d_2(3t^2+c_3t) - c_2}{6t+c_3}x - \frac{1}{6t+c_3}x^2, \\ v = {} & -\frac{1}{3}\ln|6t+c_3| + d_1 + d_2x + d_3\exp(t-x), \end{aligned}$$

where c_i and d_i are arbitrary constants. Note that this exact solution blows up in the finite time $T = -c_3/6$.

Example 2. Consider the system of PDEs

$$\begin{aligned} u_t &= [Bu_x + H_1uv_x]_x + \frac{A_1u(1-u) - vu^2}{u^2+\nu}, \\ v_t &= [Bv_x - H_2vu_x]_x + \frac{A_2u^2v}{u^2+\nu-\omega v} \end{aligned} \tag{46}$$

with $A_i, B, H_i, \omega, \nu \in \mathbb{R}$, which describes two spatially distributed populations in a predator–prey relationship with each other [5]. The spatial evolution is governed by three processes, positive taxis of predators up the gradient of prey (pursuit) and negative taxis of prey down the gradient of predators (evasion), yielding nonlinear diffusion terms, and random motion of both species (diffusion). We shall construct invariant subspaces and exact solutions of system (46) with $A_1 = \nu = \omega = 0$, which is

$$\begin{aligned} u_t &= [Bu_x + H_1 u v_x]_x - v \equiv \tilde{F}_1[u, v], \\ v_t &= [Bv_x - H_2 v u_x]_x + A_2 v \equiv \tilde{F}_2[u, v]. \end{aligned} \tag{47}$$

Here, we assume that $\Omega \subset R$ is a bounded domain and $x \in \Omega$. Our analysis leads to the following results.

(1) The vector operator $(\tilde{F}_1, \tilde{F}_2)$ preserves the invariant subspaces $W_2^1 \times W_2^2$ with $W_2^1 = W_2^2 = \mathcal{L}\{1, x\}$, which implies that system (47) has the solution of the form

$$u = c_1(t) + c_2(t)x, \qquad v = d_1(t) + d_2(t)x$$

with $c_i(t)$ and $d_i(t)$ satisfying the system of ODEs

$$\begin{aligned} c_1' &= -d_1 + H_1 c_2 d_2, \qquad & c_2' &= -d_2, \\ d_1' &= A_2 d_1 - H_2 c_2 d_2, \qquad & d_2' &= A_2 d_2. \end{aligned} \tag{48}$$

Solving the system (48) leads to the solution of system (47) with $A_2 \neq 0$

$$\begin{aligned} u &= -\frac{1}{2A_2^3}(H_1 A_2 + H_2)\exp(2A_2 t) - \frac{1}{A_2}\exp(A_2 t)x, \\ v &= \frac{1}{A_2^2} H_2 \exp(2A_2 t) + \exp(A_2 t)x. \end{aligned} \tag{49}$$

Clearly, if $A_2 < 0$, then $u \to 0$ and $v \to 0$ as $t \to \infty$.

(2) The vector $(\tilde{F}_1, \tilde{F}_2)$ admits the invariant subspaces $W_3^1 \times W_2^2$ with $W_3^1 = \mathcal{L}\{1, x, x^2\}$ and $W_2^2 = \mathcal{L}\{1, x\}$, which implies that system (47) has the solution of the form

$$u = c_1(t) + c_2(t)x + c_3(t)x^2, \qquad v = d_1(t) + d_2(t)x,$$

where $c_i(t)$ and $d_i(t)$ satisfy the system of ODEs

$$\begin{aligned} c_1' &= -d_1 + 2Bc_3 + H_1 d_2 c_2, \\ c_2' &= -d_2 + 2H_1 d_2 c_3, \\ c_3' &= 0, \\ d_1' &= A_2 d_1 - H_2(c_2 d_2 + 2d_1 c_3), \\ d_2' &= A_2 d_2 - 4H_2 c_3 d_2. \end{aligned} \tag{50}$$

Solving the system (50) yields the solution of (47)

$$\begin{aligned} u = {} & \frac{d_2^2(2c_3 H_1 - 1)(6c_3 H_1 H_2 - A_2 H_1 - H_2)}{2(6c_3 H_2 - A_2)(A_2 - 4c_3 H_2)^2}\exp(2(A_2 - 4c_3 H_2)t) \\ & + \frac{c_2 d_2(2c_3 H_1 - 1)}{2c_3(A_2 - 4c_3 H_2)}\exp((A_2 - 4c_3 H_2)t) \\ & + \frac{d_1}{2c_3 H_2 - A_2}\exp((A_2 - 2c_3 H_2)t) + 2c_3 Bt + c_1 \\ & + \left[\frac{2H_1 c_3 - 1}{A_2 - 4c_3 H_2} d_2 \exp((A_2 - 4c_3 H_2)t) + c_2\right]x + c_3 x^2, \\ v = {} & -\frac{d_2^2 H_2(2c_3 H_1 - 1)}{(4c_3 H_2 - A_2)(6c_3 H_2 - A_2)}\exp(2(A_2 - 4c_3 H_2)t) \\ & + \frac{d_2 c_2}{2c_3}\exp((A_2 - 4c_3 H_2)t) + d_1 \exp((A_2 - 2c_3 H_2)t) \\ & + d_2 \exp((A_2 - 4c_3 H_2)t)x. \end{aligned} \tag{51}$$

Here we assume that $A_2 \neq 2c_3 H_2,\ 4c_3 H_2,\ 6c_3 H_2$. If $B = 0$ and $A_2 < 2c_3 H_2$, then $u \to (c_1 + c_2 x + c_3 x^2)$ and $v \to 0$ as $t \to \infty$.

(3) The vector $(\tilde{F}_1, \tilde{F}_2)$ admits the invariant subspaces $W_3^1 \times W_3^2$ with $W_3^1 = \mathcal{L}\{1, x, x^2\}$ and $W_3^2 = \mathcal{L}\{1, x, x^2\}$. Hence the system (47) has the following solution,

$$u = c_1(t) + c_2(t)x + c_3(t)x^2, \qquad v = d_1(t) + d_2(t)x + d_3(t)x^2,$$

where $c_i(t)$ and $d_i(t)$ satisfy the system of ODEs

$$\begin{aligned} c_1' &= -d_1 + 2Bc_3 + H_1 d_2 c_2 + 2H_1 d_3 c_1, \\ c_2' &= -d_2 + 2H_1 d_2 c_3 + 4H_1 c_2 d_3, \\ c_3' &= -d_3 + 6H_1 c_3 d_3, \\ d_1' &= A_2 d_1 + 2Bd_3 - H_2 c_2 d_2 - 2H_2 c_3 d_1, \\ d_2' &= A_2 d_2 - 4H_2 c_3 d_2 - 2H_2 c_2 d_3, \\ d_3' &= A_2 d_3 - 6H_2 c_3 d_3. \end{aligned} \tag{52}$$

If $A_2 = 1,\ H_1 = H_2,\ c_2 = 0,\ d_2 = 0$ and $B = 0$, then system (52) implies that system (47) has the B-shaped solution

$$\begin{aligned} u &= \left[\frac{d_1 \exp((1-2c_3H_2)t)}{2c_3H_2 - 1} + c_1\right]\left[\frac{\exp((6c_3H_2-1)t)}{d_3(6c_3H_2-1)\exp((6c_3H_2-1)t) + 6H_2}\right]^{\frac{1}{3}} \\ &\quad + \left[-\frac{6c_3H_2 - 1}{d_3(6c_3H_2-1)\exp((6c_3H_2-1)t) + 6H_2} + c_3\right]x^2 \\ v &= d_1\left[\frac{\exp(2t)}{d_3(6c_3H_2-1)\exp((6c_3H_2-1)t) + 6H_2}\right]^{\frac{1}{3}} \\ &\quad + \frac{6c_3H_2 - 1}{d_3(6c_3H_2-1)\exp((6c_3H_2-1)t) + 6H_2}x^2. \end{aligned} \tag{53}$$

Here we assume that $c_3H_2 \neq 1/2, 1/6$. If $c_3H_2 > 1/2$, then $u \to c_1[d_3(6c_3H_2-1)]^{-1/3} + c_3x^2$ and $v \to 0$ as $t \to \infty$. Solutions such as (49), (51) and (53) play an important role in the investigation of porous medium equation [26].

Example 3. Recently, the stability of steady states of the Lokta–Volterra system with self- and cross-diffusion has been investigated by many authors (see [27] and references therein). From proposition 2.10, we know that the Lokta–Volterra system

$$\begin{aligned} u_t &= [(a_{11}u + a_{12}v + a_{13})u]_{xx} + u\big(a_1 - 4a_{11}k_1^2 u - a_{12}k_1^2 v\big), \\ v_t &= [(a_{21}u + a_{22}v + a_{23})v]_{xx} + v\big(a_2 - a_{21}k_1^2 u - 4a_{22}k_1^2 v\big) \end{aligned} \tag{54}$$

has the exact solution

$$\begin{aligned} u &= c_1(t) + c_2(t)\exp(k_1x) \in W_2^1, \\ v &= d_1(t) + d_2(t)\exp(-k_1x) \in W_2^2, \end{aligned}$$

where $W_2^1 = \mathcal{L}\{1, \exp(k_1x)\}$, $W_2^2 = \mathcal{L}\{1, \exp(-k_1x)\}$, $a_i, a_{ij}, b_i, b_{ij}, c_i, \in \mathbb{R}^+$, $k_1 \in \mathbb{R}/\{0\}$, $c_i(t)$, $d_i(t)$ satisfy the system of ODEs

$$\begin{aligned} c_1' &= -4a_{11}k_1^2c_1^2 + a_1c_1 - a_{12}k_1^2(c_2d_2 + c_1d_1), \\ c_2' &= \big(a_{13}k_1^2 + a_1\big)c_2 - 6a_{11}k_1^2c_1c_2, \\ d_1' &= -a_{21}k_1^2(c_2d_2 + c_1d_1) - 4a_{22}k_1^2d_1^2 + a_2d_1, \\ d_2' &= \big(a_{23}k_1^2 + a_2\big)d_2 - 6a_{22}k_1^2d_1d_2. \end{aligned} \tag{55}$$

Take $a_{11} = a_1 = a_{22} = b_1 = 4, a_{12} = a_{13} = a_{21} = a_{23} = k_1 = 1$. Then the autonomous system (55) becomes

$$\begin{aligned} c_1' &= -16c_1^2 - c_1d_1 - c_2d_2 + 4c_1, \\ c_2' &= -24c_1c_2 + 5c_2, \\ d_1' &= -16d_1^2 - c_1d_1 - c_2d_2 + 4d_1, \\ d_2' &= 5d_2 - 24d_1d_2, \end{aligned} \tag{56}$$

which has a stationary point $P_0 = (4/17, 0, 4/17, 0)$. Near the point P_0, we obtain the eigenvalues of the linear approximate dynamical system of (56) as follows:

$$\lambda_1 = -4, \qquad \lambda_2 = -\tfrac{60}{17}, \qquad \lambda_3 = \lambda_4 = -\tfrac{11}{17}.$$

Hence P_0 is a stable fixed point for $t \to \infty$, which implies that

$$c_1(t) \to \tfrac{4}{17}, \qquad c_2(t) \to 0, \qquad d_1(t) \to \tfrac{4}{17}, \qquad d_2(t) \to 0 \quad (t \to \infty).$$

It follows from our discussion that the exact solutions obtained by the invariant subspace method are helpful in investigating the behaviour of solutions to the considered equations.

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